ROMANOVSKIY INSTITUTE OF MATHEMATICS
NATIONAL UNIVERSITY OF UZBEKISTAN
SAMARKAND STATE UNIVERSITY

ABSTRACTS

OF THE INTERNATIONAL CONFERENCE

BRANCHING PROCESSES AND THEIR APPLICATIONS

September 18-22, 2023
Tashkent and Samarkand

V.I.Romanovskiy Institute of Mathematics, Uzbek Academy of Sciences
National University of Uzbekistan named after Mirzo Ulugbek
Samarkand State University named after Sh.R.Rashidov

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"BRANCHING PROCESSES AND THEIR APPLICATIONS"

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Branching processes

Weakly supercritical branching process in non-favorable random environment

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Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and Δ be the space of probability measures on $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ equipped with the metric of total variation. Consider the random elements $Q_1, Q_2, \ldots$, mapping $(\Omega, \mathcal{F}, \mathbf{P})$ into Δ. The sequence $\Pi = \{Q_1, Q_2, \ldots\}$ is called a *random environment.*

A sequence of non-negative integer random variables $\{Z_n,\ n \in \mathbf{N}_0\}$ is called a *branching process in random environment* (BPRE) if $Z_0 = 1$ and

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{(n)}, \qquad n \in \mathbf{N}_0,$$

where it is assumed that, conditioned on the random environment Π, the random variables $\left\{\xi_i^{(n)},\ i \in \mathbf{N}, n \in \mathbf{N}_0\right\}$ are independent and for a fixed $n \in \mathbf{N}_0$ variables $\xi_1^{(n)}, \xi_2^{(n)}, \ldots$ are identically distributed with distribution Q_{n+1}.

In the language of branching processes, Z_n is the number of particles of the nth generation, $\xi_i^{(n)}$ is the number of direct descendants of the ith particle from the nth generation. We will study the described model under the assumption that the random elements $Q_1, Q_2, \ldots$ are independent and equally distributed.

Set for $i \in \mathbf{N}$

$$X_i = \ln \varphi_i'(1), \qquad \eta_i = \frac{\varphi_i''(1)}{(\varphi_i'(1))^2}$$

(we suppose that $\varphi_1'(1), \varphi_1''(1) \in (0, +\infty)$ a.s.). We introduce the so-called *associated random walk*: $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$ for $n \in \mathbf{N}$. Note that the random vectors $(X_1, \eta_1), (X_2, \eta_2), \ldots$ are independent and identically distributed for the considered BPRE. We indicate the assumptions used in the paper regarding the random vector (X_1, η_1).

Assumption A. *The process $\{Z_i,\ i \in \mathbf{N}_0\}$ is weakly supercritical, i.e. $\mathbf{E}X_1 > 0$ and $\mathbf{E}X_1 e^{-\beta X_1} = 0$ for some $\beta \in (0, 1)$.*

Set $\gamma = \mathbf{E}e^{-\beta X_1}$. Let $F(\cdot)$ be the distribution function of a random variable X_1. We introduce the distribution function

$$F^{(\beta)}(x) = \gamma^{-1} \int_{-\infty}^{x} e^{-\beta u} dF(u), \quad x \in \mathbf{R}.$$

Assumption B. *The distribution $F^{(\beta)}(x)$ belongs to the domain of attraction of some stable law with the index $\alpha \in (1, 2]$ and is non-lattice.*

Assumption C. *For some $\varepsilon > 0$*

$$\mathbf{E}\left[\left(\ln^+ \eta_1\right)^{\alpha+\varepsilon} \exp\left(-\beta X_1\right)\right] < +\infty.$$

We introduce a random process Y_n:

$$Y_n(t) = \frac{Z_{\lfloor nt \rfloor}}{\exp\left(S_{\lfloor nt \rfloor}\right)}, \quad t \in [0,1); \qquad Y_n(1) = Z_n.$$

Let the symbol $\Rightarrow$ denote convergence in the sense of finite-dimensional distributions. We formulate the main results. Set for $n \in \mathbf{N}$

$$M_n = \max_{1 \le i \le n} S_i.$$

The associated random walk $\{S_n\}$ of the supercritical BPRE has a positive drift and therefore S_n and M_n tend to $+\infty$, as $n \to \infty$. In this paper, a weakly supercritical branching process is considered in non-favorable random environment of two types, when either $M_n < 0$ or $S_n \le u$, where u is a positive constant. For the random process $\{Y_n(t), t \in [0,1]\}$, the following functional limit theorems are proved in the case of non-favorable environments of these types.

Theorem 1. *If assumptions* **A**, **B** *and* **C** *are valid, then for any $u \in [-\infty, 0)$, as $n \to \infty$,*

$$\{Y(t),\ t \in [0,1]\ |\ M_n < 0,\ S_n \ge u\} \Rightarrow \{U_u(t),\ t \in [0,1]\},$$

where $\{U_u(t),\ t \in [0,1]\}$ is a random process with non-negative constant trajectories on the interval $(0,1)$, and the probability of the event $\{U_u(t) > 0\}$ is positive for $t \in (0,1)$; the random variable $U_u(1)$ takes values from $\mathbf{N}_0$ and the probability of the event $\{U_u(1) > 0\}$ is positive.

Set

$$\mathcal{T}_n = \max\{i : S_i = \max(0, M_n),\ 0 \le i \le n\},$$

that is, $\mathcal{T}_n$ is the last moment when the maximum of the random walk $S_0, \ldots, S_n$ is attained.

Theorem 2. *If assumptions* **A**, **B** *and* **C** *are valid, then for any $u \in (0, +\infty)$, as $n \to \infty$,*

$$\{Y_n(t),\ t \in [0,1]\ |\ \mathcal{T}_n = n,\ S_n \le u\} \Rightarrow \{V_u(t),\ t \in [0,1]\},$$

where $\{V_u(t),\ t \in [0,1]\}$ is a random process with non-negative constant trajectories on the interval $(0,1)$, and the probability of the event $\{V_u(t) > 0\}$ is positive for $t \in (0,1)$; the random variable $V_u(1)$ takes values from $\mathbf{N}_0$ and the probability of the event $\{V_u(1) > 0\}$ is positive.

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Limit theorem for a subcritical branching process with continuous time and migration

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Keywords: branching process, limit theorem, migration, subcritical.

Branching process are mathematical models of many physical, biological, genetic, demographic and other processes. Since third- party factors often exist, there is a need to study different modifications of this process. Among them are branching processes, with immigration, emigration, or a combination of processes, namely processes with migration for the case of discrete and conditions time.

An important feature of the branching process is the generating function. In the classical case, for processes with continuous time, it is obtained from the differential equation.

In the case of the branching processes with immigration, the derivation of the differential equation and finding its solution in [1], where the process is defined as a process with two types of particles.

In the case of the process of emigration, Formanov Sh.K. and Kaverin S.V. found the form of a differential equation and the solution of this equation without detained inference is shown in [3],[4].

The main results for branching processes with discrete time and different regimes of immigration and emigration are described in [5].

The limit distribution theorem for the classical branching process with continuous time is proved in [2]. In this work we consider a more general model of the branching processes with migration and continuous time [6].

Immigration, emigration and evolution occur at random moments of time and are determined by the intensity of the transition probabilities.

The form of a generating function for a branching process with migration and continuous time and the Kolmagorov system of equation held for the transition probabilities of the process are found in [6].

Consider a Markov branching process with one type of particles and migration $\mu(t)$, $t \in [0,\infty)$. Let $\mu(t)$ denote the number of particles at the time $t \in [0,\infty)$.

We suppose, that at the time $t = 0$, the process starts with one particle in the system $\mu(0) = 1$.

The process $\mu(t), t \in [0,\infty)$ then $\Delta t \to 0$ is given by transition probabilities $P\{\mu(t+\Delta t) = j/\mu(t) = i\} = P_{ij}(t)$, which are expressed by intensity of reproduction particle $p_k(k = 0, 1, ...)$, the intensity of immigration $q_k(k = 0, 1, ...)$, and the intensity of emigration $r_n(n = \overline{0, m})$ [7].

p_k, q_k, r_n satisfy the conditions

$$p_k \geq 0,\ k \neq 1,\ p_1 < 0,\ \sum_{k=0}^{\infty} p_k = 0,$$

$$q_k \geq 0,\ k \neq 0,\ q_0 < 0,\ \sum_{k=0}^{\infty} q_k = 0,$$

$$r_n \geq 0,\ n = \overline{1,\ m},\ r_0 < 0,\ \sum_{k=0}^{m} r_k = 0,$$

We introduce the following notation

$$f(s) = \sum_{n=0}^{\infty} p_n s^n,\ |s| \leq 1,$$

$$g(s) = \sum_{n=0}^{\infty} q_n s^n,\ |s| \leq 1,$$

$$r(s) = \sum_{n=0}^{m} q_n s^{-n},\ 0 \leq |s| \leq 1,$$

We consider the subcritical branching process and find limiting distribution

Theorem. If $a_0 = f'(1) < 0$, $a_1 = g'(1) < \infty$, $a_2 = r'(1) < \infty$ and $\int\limits_0^{\infty} M\mu(x)dx < \infty$, then limiting distribution $\mu(t)$ exists

$$\lim_{t\to\infty} P\{\mu(t) = n\} = P_n^*.$$

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Limit theorems for the branching process with decreasing immigration

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Keywords: Branching random process, state-dependent immigration, slowly varying function, generating function.

Let μ_n be a number of particles of the Galton-Watson (G-W) branching process at the moment n ($n = 0, 1, \ldots, \mu_0 = 1$) with the generating function (g.f.)

$$F(x) = \sum_{j=0}^{\infty} p_j x^j, \quad p_j = P\{\mu_1 = j\}, \quad j = 0, 1, ..., \quad |x| \leq 1.$$

The zero state is absorbing for the process μ_n , that is, if $\mu_N = 0$ for some $N > 0$, then $\mu_n = 0$ for all $n > N$. In [1] J.H.Foster considered G-W process modified to allow immigration of particles whenever the number of particles is zero. If $\mu_n = 0$, then, at the moment n, ξ_n particles immigrate to the population, where the number of particles evolves by the law of the G-W process with g.f. $F(x)$.

The asymptotic behavior of branching processes with state-dependent immigration were studied by many authors (see [1]–[3]).

We consider the case when immigration takes place as $\mu_n = k$, $0 \leq k \leq m$, where m is some nonnegative integer. Assume that the intensity of the immigration decreases tending to 0, when the number of descendents increases. Limit theorems for such processes have been studied in [4],[5],[6].

Thus, the immigration is given with g.f.

$$g_{k,n}(x) = \sum_{j=0}^{\infty} q_{kj}(n) x^j, \quad |x| \leq 1, \quad k = 0, 1, ..., m, \quad q_{kj}(n) \geq 0,$$

$$\sum_{j=0}^{\infty} q_{kj}(n) = 1, \qquad n = 0, 1, 2, ... \quad .$$

Let$\{Z_n;\, n = 0, 1, ...\}$ be a number of particles of this process at the moment n.
Suppose, that

$$F(x) = x + (1 - x)^{1+\nu} L(1 - x)$$

where $0 < \nu \leq 1$ and $L(x)$ is a slowly varying function (s.v.f.) as $x \to 0$.

Introduce the function

$$M(n) = \sum_{k=1}^{n} \frac{N(k)}{k^{1/\nu}},$$

where $N(x)$ is a s.v.f. as $x \to \infty$ such that $\nu N^{\nu}(x) L(N(x)/x^{1/\nu}) \to 1$.

Denote

$$\alpha_n = \max_{0 \leq k \leq m} g'_{k,n}(1) \qquad \beta_n = \max_{0 \leq k \leq n} g''_{k,n}(1),$$

$$Q_1(n)=\alpha_n\sum_{k=0}^{n}(1-F_k(0)),\qquad Q_2(n)=(1-F_n(0))\sum_{k=0}^{n}\alpha_k$$

where $F_0(x)=x,\quad F_{n+1}(x)=F\left(F_n(x)\right)$.
We suppose that

$$\underset{n}{Sup}\,\alpha_n<\infty\,,\qquad \underset{n}{Sup}\,\beta_n<\infty\,,$$

$$0<\alpha_n\to 0\,,\qquad \beta_n\to 0\,,\qquad n\to\infty.$$

We consider the case $\nu=1,\quad M(n)\to\infty$ as $n\to\infty$.

Theorem 1. Let

$$\alpha_n\sim\frac{l(n)}{n^r},\ \beta_n=o\left(Q_1(n)\right),\quad n\to\infty,$$

where $0\leq r<1$ and $l(n)$ is a s.v.f. as $n\to\infty$. Then for all $0\leq x\leq 1$

$$\lim_{n\to\infty}P\left\{\frac{M(Z_n)}{M(n)}<x/Z_n>0\right\}=x.$$

Theorem 2. Let $\alpha_n\sim\frac{N(n)}{n},\ \beta_n=o\left(Q_1(n)\right),\quad n\to\infty$ and $\lim\limits_{n\to\infty}\frac{Q_1(n)}{Q_2(n)}=a,\quad a\geq 0.$
Then

a) for all $0<x<1$,

$$\lim_{n\to\infty}P\left\{\frac{M(Z_n)}{M(n)}<x/Z_n>0\right\}=\frac{ax}{1+a};$$

б) for $x\geq 0$,

$$\lim_{n\to\infty}P\left\{\frac{N(n)Z_n}{n}<x/Z_n>0\right\}=\frac{a}{1+a}+\frac{1-e^{-x}}{1+a}.$$

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Approximation of size-density dependent branching processes

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Consider the branching process (Z_n) generated by the recursion

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{n,j}, \quad n = 1, 2, ...$$

subject to the initial condition Z_0. The integer valued random variables $\xi_{n,j}$ are conditionally i.i.d. given the previous generations $\mathcal{F}_{n-1} = \sigma\{\xi_{m,j} : j \in \mathbf{N}, m \leq n\}$ and their common distribution depends on the size-density $\overline{Z}_{n-1} = K^{-1} Z_{n-1}$ with respect to the large parameter K:

$$\mathsf{P}(\xi_{n,j} = \ell | \mathcal{F}_{n-1}) = p_\ell(\overline{Z}_{n-1}), \quad \ell \in \mathbf{Z}_+, \tag{1}$$

where $p_\ell(\cdot) \geq 0$ are some functions, $\sum_{\ell=0}^\infty p_\ell(x) = 1$. If the offspring conditional mean $m(x) = \sum_{\ell=0}^\infty \ell p_\ell(x)$ is a decreasing function with $\rho := m(0) > 1$ and $m(1) = 1$, this random process can be viewed as a basic model for populations on a habitat with the carrying capacity K. Its typical trajectory started from $Z_0 \ll K$ grows rapidly until reaching the capacity region, where it fluctuates for a very long period of time until eventual extinction, see e.g. [1].

This lifecycle can be described by the limit behavior of the size-density process $\overline{Z}_n$ satisfying

$$\overline{Z}_n = f(\overline{Z}_{n-1}) + \frac{1}{K} \sum_{j=1}^{Z_{n-1}} \big(\xi_{n,j} - m(\overline{Z}_{n-1})\big), \quad n = 1, 2, ...$$

where $f(x) := xm(x)$. The last term is of order $O_{\mathsf{P}}(K^{-1/2})$ and hence, under mild conditions,

$$\max_{m \leq n} \big|\overline{Z}_n - x_n\big| \xrightarrow[K\to\infty]{\mathsf{P}} 0, \quad \forall n \in \mathbf{Z}_+,$$

where the limit sequence x_n solves the recursion

$$x_n = f(x_{n-1}), \quad n = 1, 2, ... \tag{2}$$

subject to $x_0 = \lim_{K\to\infty} \overline{Z}_0$. The stochastic fluctuations of $\overline{Z}_n$ about the deterministic limit converge to a Gaussian process $V = (V_n)$ in distribution: $\sqrt{K}(\overline{Z}_n - x_n) \xrightarrow[K\to\infty]{d} V_n$ where V_n satisfies the recursion, [2],

$$V_n = f'(x_{n-1}) V_{n-1} + \sqrt{x_{n-1}\sigma^2(x_{n-1})} W_n, \quad n = 1, 2, ...$$

with $N(0,1)$ i.i.d. random variables W_n's. Large Deviations analysis [3] reveals that the time to extinction grows exponentially with K.

This theory gives an adequate picture of the population dynamics, when its initial size is relatively large, i.e., proportional to the capacity so that $x_0 := \lim_{K\to\infty} \overline{Z}_0 > 0$. In the compliment case of Z_0 being small, e.g. $Z_0 = 1$, the recursion (2) is started from $x_0 = 0$ and, consequently, the limit sequence degenerates to $x_n = f^n(x_0) = 0$ for all $n \geq 0$. Nevertheless, when K is large but finite, some of the trajectories manage to escape the early extinction and still reach the capacity region, albeit at a much later time, logarithmic in K. In this talk I will present a new type of limit theorems which capture such a delayed emergence, [4]-[6].

On a suitable probability space, (Z_n) can be coupled with an auxiliary Galton-Watson process (Y_n) with the offspring mean ρ, started from $Y_0 = Z_0$. The sequence $\rho^{-n} Y_n$ is a convergent martingale with the a.s. limit $W = \lim_{n\to\infty} \rho^{-n} Y_n$. Moreover, under certain technical conditions on f, the sequence of the scaled iterates can be shown to converge to a limit:

$$H(x) := \lim_{n\to\infty} f^n(x/\rho^n).$$

Theorem 1. [6] Let $n_1 := n_1(K) = [\log_\rho K]$ then

$$\overline{Z}_{n_1} - H\Big(W\rho^{-\{\log_\rho K\}}\Big) \xrightarrow[K\to\infty]{\mathsf{P}} 0, \tag{3}$$

where $\{x\} = x - [x]$ for $x > 0$.

In particular, when K is an integer power of ρ, this result implies that the distribution of the size-density process at times of order $\log_\rho K$ is close to the distribution of the random variable $H(W)$. The proof of Theorem 1 is based on the long known heuristics [7], [8] according to which a size dependent population behaves initially as the Galton-Watson branching process and, if it manages to avoid extinction at this early stage, it continues to grow towards the carrying capacity following an almost deterministic curve. This suggests to approximate $\overline{Z}_n$ by means of $\overline{Y}_n$ for $n \leq n_c$ where $n_c = [\log_\rho K^c]$ with some $c \in (0,1)$ and by means of $f^{n-n_c}(\overline{Y}_{n_c})$ for $n > n_c$. This approximation at time n_1 produces the limit of Theorem 1 if c is taken to be any number in $(\frac{1}{2}, 1)$.

This construction depends on a free parameter c and a close inspection of the proof reveals that the best rate of convergence in (2) is $O_{\mathsf{P}}(K^{-1/8} \log K)$ and it is achieved with $c = \frac{5}{8}$. On the other hand, in the special case when Z_n is a Galton-Watson process, i.e. the probabilities in (1) are constant with resect to $\overline{Z}_n$, Heyde's CLT [9] implies that the rate of convergence is $O_{\mathsf{P}}(K^{-1/2})$. In our recent paper [10] we show how the above construction can be modified to produce an almost optimal rate of $O_{\mathsf{P}}(K^{-1/2} \log K)$.

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Consistent estimation for population-size-dependent branching processes

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Keywords: Branching process; population-size-dependence; almost sure extinction; inference; carrying capacity; Q-process.

We consider population-size-dependent branching processes (PSDBPs) which eventually become extinct with probability one. These processes are designed to capture the dynamics of endangered populations living in restricted habitats with a carrying capacity. We derive maximum likelihood estimators (MLEs) for the mean number of offspring born to individuals when the current population size is $z \geq 1$, based on a single trajectory of population size counts. Because these processes become extinct with probability one, we show that the MLEs do not satisfy the classical consistency property (C-consistency). This leads us to define the concept of Q-consistency, and we prove that the MLEs are Q-consistent and asymptotically normal. Using these MLEs, we then propose a new class of weighted least-squares C-consistent estimators for parametric PSDBPs with logistic growth.

Our results are motivated by conservation biology, where endangered populations are often studied because they are still alive, thereby inducing an observation bias. Through simulated examples, we show that our C-consistent estimators generally reduce this bias, leading to improved estimates for important quantities such as a habitat's carrying capacity. We apply our methodology to estimate the carrying capacity of the Chatham Island black robin −a population that was reduced to a single breeding female in the 1970's, which has since recovered but is yet to reach the island's carrying capacity

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Asymptotic Shape of Branching Random Walks on Periodic Graphs

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Keywords: branching random walk, asymptotic shape, propagation front, supercritical regime, light tails.

A multitude of publications devoted to different models of branching random walk have appeared within the last decade (see, e.g., [1]–[5]). Any model of branching random walk (BRW) comprises two random mechanisms. The first one accounts for splitting and death of particles and the second one for their random moving in space. Various combinations of the two arrangements lead to different models of BRW. These models are not only of theoretical interest but also have numerous applications in biology, chemical kinetics, statistical physics, homopolymers theory, queueing theory, etc. (see, e.g., [6]–[8]).

A special place in the theory of BRW is occupied by catalytic models (see, e.g., [9] and [10]). The term “catalytic” means that there are several points of the space where the catalysts are located and just these catalysts make a particle reaching such a location either split or die there. At a point without a catalyst a particle may walk only. Accordingly, the points containing catalysts are called points of catalysis or sources of branching and death of particles. The case of a single source of branching was investigated in many papers (see, e.g., [11] and [12]). The description of the spread of particles population in catalytic BRW with an arbitrary finite number of branching sources on Z^d was initiated in papers [13] and [14], and the study was accomplished in a series of papers by the author, we refer, e.g.,to [15]–[17]. The case of an infinite set of branching sources having periodical structure, i.e. BRW on periodic graphs, was treated for the first time in [18] and [19]. Those papers describe the asymptotic behavior, with respect to growing time, of means of some functionals related to BRW on periodic graphs.

In the present talk, in contrast to [18] and [19], we consider the spread almost sure of particles population in BRW on periodic graphs and study the asymptotic behavior (as time t goes to infinity) of the normalized cloud of particles existing at time t. We stipulate that the regime of branching is supercritical and the jumps of a random walk have light tails. Under these assumptions the instant positions of all the particles at time t are normalized by factor t^{-1} before letting $t \to \infty$. The corresponding set of the normalized particles positions at time t is denoted by $\mathcal{P}_t$. We establish that

$$\Delta\left(\mathcal{P}_t, \mathcal{P}\right) \to 0 \quad \text{a.s. on event } \mathcal{S} \text{ of population non-degeneracy}, \quad \text{as} \quad t \to \infty,$$

where $\Delta(D, F)$ is the Hausdorff distance between sets D and F belonging to R^d. The limiting set $\mathcal{P} \subset R^d$ is called the asymptotic shape of the BRW. We prove that, in the mentioned case of periodical structure, the arising set $\mathcal{P}$ is compact and convex. Moreover, we also provide an explicit formula to describe it in terms of the level sets of the Perron roots of a family of some parametric quasi-nonnegative matrices. Each element of such a matrix is the Laplace transform of the intensity measure of a specified point process.

This means that the particles population spreads asymptotically linearly in time and the shape of random cloud of particles is approximated by $t\mathcal{P}$ as $t \to \infty$. Our main result also shows that, with probability one, the cloud of normalized particles not only becomes asymptotically close to the boundary of $\mathcal{P}$ but fills the whole set $\mathcal{P}$. The assumption of supercriticality guarantees the positive probability of the event $\mathcal{S}$ of the population survival. On the opposite event the problem of the rate of the population spread is meaningless because of the population degeneracy. The assumption of light tails of the random walk jumps leads to the indicated normalizing factor. We also tackle other conditions concerning the behavior of random walk tails (e.g., the tails are regularly varying or the jump has a semi-exponential distribution). Such conditions, effecting the population spread, were employed in [16] and [17] for BRW with finitely many sources of branching.

Note that in papers [18] and [19] the authors apply spectral theory of operators to analyze BRW on periodic graphs. For this reason they additionally assume the symmetry of the random walk. However, we use other methods based on consideration of our BRW in the scope of a general BRW with finitely many types of particles. This approach allows to avoid additional assumption on symmetry of the random walk and permits to consider the random walk having, for example, a drift. Our proof introduces specified auxiliary stochastic process and essentially relies on paper [20] devoted to general BRW.

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Recent results on eigenvalues for branching processes and related fields

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Law of large number type result for branching processes is related to the long time behavior of some non-conservative semi-group. We will present some new results for such semigroup in the particular case of non-local branching and deterministic dynamics between branching. These results are based on eigen-problem associated to some integro-differential operators. The techniques will be related to Doeblin and Lyapunov arguments. Finally several examples coming from applications in biology will be developed (evolution models, growth fragmentation...).

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Lower Large Deviations of Strongly Supercritical Branching Process in Random Environment with Geometric Number of Descendants: Local Asymptotics

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Keywords: branching processes, random environment, random walk, Cramer's condition, large deviations, local theorems.

Let $\boldsymbol{\eta} = (\eta_1, ...)$ be a sequence of independent identically distributed (i.i.d.) random variables (r.v.). Let $\{\phi_y\}_{y\in\mathbb{R}}$ be a family of generating functions. For fixed $\boldsymbol{\eta}$ we consider independent random variables $X_{i,j}, j \in \mathbb{N}$, with generating functions $\phi_{\eta_i}, i \in \mathbb{N}$, for each $i \in \mathbb{N}$. We define the branching process $(Z_n, n \geq 0)$ in random environment $\boldsymbol{\eta}$ (BPRE) by the following formula:

$$Z_0 = 1, \quad Z_{n+1} = X_{n+1,1} + \dots + X_{n+1,Z_n},\ n \geq 0.$$

Let $\xi_i = \ln \phi'_{\eta_i}(1)\,, \mathbf{E}\xi_i = \mu$. The random walk $S_n = \xi_1 + \dots + \xi_n, n \geq 1$, is called the associated random walk.

We suppose that the step of the associated random walk has positive mean μ and satisfies left-hand Cramer's condition $\mathbf{E}\exp(h\xi_i) < \infty$ as $h^- < h < 0$, where h^- is some parameter, and that for every i, j r.v. $X_{i,j}$ have geometric distribution. Also we suppose that ξ_i is non-lattice r.v. Under these assumptions we consider local probabilities of lower deviations for BPRE. In other words, we study the local probabilities $\mathbf{P}\left(Z_n = \lfloor \exp\left(\theta n\right) \rfloor\right)$ as $n \to \infty$ and $\theta \in (\max(m^-, 0); \mu)$ for some constant m^-.

Large deviations of BPRE are well studied. The asymptotic behavior of $\mathbf{P}(Z_n > \exp(\theta n))$ as $n \to \infty$, where $\theta > \mu$, for BPRE with geometric distribution of the number of descendants of one particle (BPREG) was studied by Kozlov ([1], [2]). In the general case, both the large devation principle ([3]) and the exact asymptotics ([4], [5]]) were obtained. For the probabilities of lower deviations $\mathbf{P}(1 \leq Z_n < \exp(\theta n))$, where $\theta < \mu$, only logarithmic asymptotic representation was obtained ([6]).

In this report we consider local lower large deviations of BPREG. In other words, we study $\mathbf{P}(Z_n = k)$, where $k(n) = k \in \mathbb{N}$. We assume that $\theta(n) = \theta := \ln k/n$ belongs to some segment $[\theta_1; \theta_2] \subset (\max(m^-, 0); \mu)$. Under these assumptions we introduce two deviation zones: the first deviation zone — $(\max(m(-1), 0); \mu)$, and the second deviation zone — $(\max(m^-, 0); m(-1))$, where $m(-1)$ is some parameter. We show that

$$\mathbf{P}\left(Z_n = k\right) = \frac{1 + o(1)}{\sqrt{2\pi n}\sigma\left(h_\theta\right)} e^{-\Lambda(\theta)n - \theta n} \Gamma\left(1 + h_\theta\right) \mathbf{E}\widetilde{V}_\infty^{h_\theta - 1}$$

as $n \to \infty$ uniformly in $\theta \in [\theta_1; \theta_2]$ where θ_1, θ_2 belongs to the first deviation zone, and

$$\mathbf{P}\left(Z_n = k\right) = \left(1 + o(1)\right) R^n(-1) \mathbf{E}\widehat{V}_\infty^{-2}$$

as $n \to \infty$ uniformly in $\theta \in [\theta_1; \theta_2]$ where θ_1, θ_2 belongs to the second deviation zone ([8]). Here $\widetilde{V}_\infty$ and $\widehat{V}_\infty$ are some r.v. and h_θ, $R(n)$ and $\Lambda(\theta)$ are some functions. Moreover, we prove that

$$\mathbf{P}\left(Z_n = k\right) = \left(1 + o(1)\right) \times$$
$$\times \mathbf{E}\widetilde{V}_\infty^{h_\theta - 1} e^{-\Lambda(\theta)n - \theta n} \exp\left(\frac{\sigma^2(h_\theta) n (1 + h_\theta)^2}{2}\right) \left(1 - \Phi\left(\frac{\sqrt{n}(\theta - m(-1))}{\sigma(h_\theta)}\right)\right)$$

as $n \to \infty$ uniformly in $\theta \in [\theta_1(n); \theta_2(n)] \subset (\max(m^-, 0); \mu)$ such that $\theta_1(n) \to m(-1)$ and $\theta_2(n) \to m(-1)$ as $n \to \infty$. Thus, we obtain the asymptotic representation of $\mathbf{P}\left(Z_n = k\right)$ as $n \to \infty$ that is uniform in $\theta \in [\theta_1; \theta_2] \subset (\max(m^-, 0); \mu)$.

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On reduced processes starting from a large number of particles

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Keywords: subcritical and critical branching processes, reduced process, weak convergence.

Let $\{Z_k, k \geq 0\}$ the Galton-Watson branching process (see e.g., [1]) in which the number of direct descendants of one particle have a generating function $f(s)$, $0 \leq s \leq 1$.

Denote by $Z(m,n)$ the number of particles in the moment m $(m \leq n)$ in the process $\{Z_k, k \geq 0\}$, whose descendants exit at the moment n. The random process $\{Z(m,n), 0 \leq m \leq n\}$ is called the reduced process generated by the process $\{Z_k, k \geq 0\}$. The reduced process $\{Z(m,n), 0 \leq m \leq n\}$ is called subcritical, critical and supercritical if $f'(1) < 1$, $f'(1) = 1$ and $f'(1) > 1$ respectively. Reduced subcritical processes for Galton-Watson processes were introduced by Fleischmann and Prehn [2] . Fleischmann and Sigmund-Schultze [3] proved a functional limit theorem (under the assumption $Z_n > 0$) in which the convergence of reduced critical processes to the Yule process is established. Liu and Vatutin [4] proved conditional limit theorems (under the assumption $0 < Z_0 \leq \psi(n)$) for reduced critical processes starting with a single particle and with a finite variance in the number of direct descendants of a single particle.

In this report, we propose limit theorems for subcritical and critical reduced processes $\{Z(m,n), 0 \leq m \leq n\}$ in the case when $Z_0 = \varphi(n)$ with probability 1, where $\varphi(n)$ such that $\varphi(n) \sim n$ or $\varphi(n) = o(n)$ when $n \to \infty$.

We present one of our results.

Theorem. Let for a reduced critical process

$$0 < f''(1) = \sigma^2 < \infty$$

and with probability 1 $Z_0 = [xn]$, where is $x > 0$ a fixed number, the sign $[a]$ means the integer part of the number a. Then for any $t \in [0,1)$ the asymptotic relation holds

$$E\left[s^{Z([nt],n)} / \, Z_0 = [xn]\right] \to e^{-\frac{2x}{\sigma^2} \cdot \frac{1-s}{1-st}}, \quad n \to \infty. \tag{1}$$

If $t = 0$, then in the limit (1) a Poisson distribution arises with the parameter $\frac{2x}{\sigma^2}$. So the distribution of the number $Z_0 = [xn]$ of initial particles that have a descendant in the n-th generation at the $n \to \infty$ approaches the Poisson law. This effect is understandable due to the similarity of the law of development of initial particles.

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Explosions and dualities in logistic continuous state branching processes

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Keywords: branching processes with competition, stochastic duality, explosion, local time
Call Z the process of the size of a random continuous-state population in which classical reproduction of individuals, governed by a branching mechanism Ψ, (with possibly gigantic number of offsprings !) is counterbalanced by a quadratic competition drift term: " $-\frac{c}{2}Z_t^2dt$ ", somehow resulting from duel fights between individuals: two individuals fight and one kills the other (logistic competition term). These processes were introduced by A. Lambert in 2005, see [3], who baptized them Logistic CSBPs (LCSBPs).

I will explain in the talk two duality relationships and some of their consequences. By exploring first a Laplace duality between LCSBPs Z and certain generalized Feller diffusions U:

$$Z \overset{\textbf{Laplace dual}}{\longleftrightarrow} U$$

we will see that the boundary 0 of the diffusion process U is regular *absorbing* if and only if the boundary ∞ of the LCSBP Z is regular *reflecting* (the process leaves and returns immediately to ∞ without spending in it a second). Explicit necessary and sufficient conditions on the branching mechanism and the competition term are found for this to hold and the Laplace duality is used to build the extension of the LCSBP, after the first explosion, when it exists. This part is taken from [1].

In order to study further the process past explosion and for instance to identify its local time, we are going to introduce another auxiliary process, V, obtained as the Siegmund dual of the process U:

$$U \overset{\textbf{Siegmund dual}}{\longleftrightarrow} V$$

The process V is a certain bi-dual process of Z and many nice connections between Z and V can be explored, see [2].

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Large Deviation for Supercritical Controlled Branching Processes

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Keywords: controlled branching processes; supercritical case; large deviations

The topic of large deviation estimates plays an important role for researching many questions in statistics. In particular, in the field of the branching processes the study of large deviations for standard Bienaymé-Galton-Watson branching processes (BGWPs), $\{Z_n\}_{n\geq 0}$, was initiated in [1] and [2] and investigated in detail in [3] and [4].

Among others results, the large deviation behavior of the statistic $R_n = Z_{n+1}Z_n^{-1}$ has been studied. This statistic has been used in the estimation of the amplification rate in a quantitative polymerase chain reaction (PCR) experiment where only Z_n and Z_{n+1} are observed. For a BGWP, it is known that $\{Z_{n+1}Z_n^{-1}\}_{n\geq 0}$ converges almost surely to m, denoting by m the offspring mean, when $m > 1$ (supercritical case) on the non-extinction set. In the quoted papers previously, the rate of decay as $n \to \infty$ of $P\left(\left|\frac{Z_{n+1}}{Z_n} - m\right| > \epsilon\right)$ was studied. In particular, the reproductive property of a BGWP was used to link the large deviations of $Z_n^{-1}Z_{n+1}$ with the rates of convergence of the generating function via the use of the harmonic moments of Z_n. The harmonic moments of these processes was carried out in [5].

A control branching process (CBP) is a generalization of Byenaimé–Galton–Watson processes where at each generation the number of progenitors is randomly chosen through a random control function.

In this talk we present large deviation results for supercritical controlled branching processes under an assumption on the exponential moments or polynomial moments of the offspring distribution and also based on the asymptotic behaviour of the harmonic moments of the generation sizes.

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Conditional central limit theorem for critical and subcritical branching random walk

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Consider a branching random walk on $\mathbb{R}$. Let $Z_n(A)$ be the number of the individuals in the n-th generation located in $A \in \mathcal{B}(\mathbb{R})$, and $N_n := Z_n(R)$ denote the size of n-th generation, which is a Galton-Watson process. It is well know that when $\mathbf{E}N_1 = m > 1$, the central limit behavior have been considered by Kaplan and Asmussen (1976) and Biggins (1990). For the critical and subcritical case, we obtained the behavior as follows,

(1) when $\mathbf{E}N_1 = m = 1$, we prove that, under some conditions, for all $x \in \mathbb{R}$, as $n \to \infty$,

$$\mathcal{L}\left(\frac{Z^{(n)}(-\infty, \sqrt{n}x]}{n} \,\middle|\, N_n > 0\right) \Longrightarrow \mathcal{L}\left(Y(x)\right),$$

where $\Rightarrow$ means convergence in law and $Y(x)$ is a random variable whose distribution is specified by its moments.

(2) when $0 < \mathbf{E}N_1 = m < 1$, we prove that for $x \in \mathbb{R}$, under some conditions, as $n \to \infty$,

$$\mathcal{L}\left(Z_n((-\infty, \sqrt{n}x]) \mid N_n > 0\right) \Longrightarrow \mathcal{L}(\xi \mathbf{1}_{\{\mathcal{N} \le x\}}),$$

where $\Rightarrow$ means convergence in law, ξ is the Yaglom limit of the subcritical Galton-Watson process $\{N_n; n \geq 0\}$ conditioned on non-extinction, $\mathcal{N}$ is a standard normal random variable and independent of ξ .

This is a joint work with Shengli Liang and Dan Yao.

Keywords: *branching random walk, subcritical Galton-Watson process, critical Galton–Watson process, reduced process, Yaglom's limit, conditional central limit theorem.*

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On explicit expression of the Generating Function of Invariant Measures of Critical Galton-Watson Branching Systems

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Keywords: *Galton-Watson Branching System, Generating functions, Slow variation, Basic Lemma, Transition probabilities, Invariant measures, Limit theorems, Convergence rate.*

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Consider an ordinary Galton-Watson Branching (GWB) system with a state space $\mathcal{S}_0 \subset \mathbb{N}_0$ and an offspring law $\{p_j, j \in \mathcal{S}_0\}$. Let $Z(n)$ be the population size at the moment $n \in \mathbb{N}_0$. The stochastic system $\{Z(n)\}$ forms a reducible, homogeneous and discrete-time Markov chain whose state space consists two classes: $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\mathcal{S} \subset \mathbb{N}$, therein $\{0\}$ is an absorbing state, and $\mathcal{S}$ is the class of possible essential communicating states. The offspring law $\{p_k, k \in \mathcal{S}\}$ fully defines a structure of the GWB system. In fact, we observe that an appropriate probability generating function (GF) $\mathsf{E}\left[s^{Z(n)} \mid Z(0) = i\right] = \left[f_n(s)\right]^i$ for all $s \in [0,1)$, where the GF $f_n(s) = \mathsf{E}_1 s^{Z(n)}$ is n-fold functional iteration of GF

$$f(s) := \sum_{k \in \mathcal{S}_0} p_k s^k.$$

Denoting q be the smallest root of the fixed-point equation $f(s) = s$ for $s \in [0,1]$, we recall that $f_n(s) \to q$ as $n \to \infty$ uniformly in $s \in [0,r]$ for any fixed $r < 1$. So, the GWB system is a discrete dynamic system generated by the GF $f(s)$ and with the fixed point q, which is an extinction probability of a trajectory of the system initiated by a single founder; see [1, Ch. I].

We consider a case when the offspring GF $f(s)$ for $s \in [0,1)$ admits the following form:

$$f(s) = s + (1-s)^{1+\nu} \mathcal{L}\left(\frac{1}{1-s}\right), \qquad [f_\nu]$$

where $0 < \nu < 1$ and $\mathcal{L}(*)$ slowly varies [2] at infinity. Assumption $[f_\nu]$ implies that the per-capita offspring mean $m := \sum_{j \in \mathcal{S}} j p_j = f'(1-) = 1$ and $f''(1-) = \infty$, so that our system is critical type with infinite variance. In the case when the slowly varying at zero function $L(*)$ replaces $\mathcal{L}(*)$ in $[f_\nu]$, Slack [4] has shown that there exists an invariant measure whose GF $U(s)$ has the following local expression:

$$U(s) \sim \frac{1}{\nu (1-s)^\nu L(1-s)} \qquad \text{as} \quad s \uparrow 1.$$

In this report we provide an alternative argument against Slack's one and we obtain the global expression for all $s \in [0,1)$ of the function $U(s)$ and its derivative. Let

$$\mathcal{V}(s) := \frac{1}{\nu \Lambda(1-s)} \qquad \text{and} \qquad J(s) := \frac{1 - f'(s)}{\Lambda(1-s)} - 1,$$

where $\Lambda(y) := y^{\nu}\mathcal{L}(1/y)$.

Theorem 1. If $p_0 > 0$ and the condition $[f_\nu]$ is satisfied, then

(i) the GF $U(s)$ has the following form:

$$U(s) = \mathcal{V}(s) - \mathcal{V}(0);$$

(ii) the derivative $U'(s)$ has the following expression:

$$U'(s) = J(s)\frac{\mathcal{V}(s)}{1-s}.$$

Remark. The function $U(s)$ admits the power series expansion $U(s) = \sum_{j\in\mathcal{S}} u_j s^j$, where $u_j = \sum_{k\in\mathcal{S}} u_k P_{kj}(1)$ and $\sum_{k\in\mathcal{S}} u_k p_0^k = 1$; see [4, Lemma 4]. Then it follows that

$$u_1 = U'(0) = \frac{J(0)}{\nu p_0} = \frac{1-p_0-p_1}{\nu p_0^2}.$$

The assertions of the Theorem improve the corresponding results from [3].

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On refinement of some limit theorems for the noncritical Galton-Watson Branching Systems

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Let $Z(n)$ be a population size at the moment $n \in \mathbb{N}_0$ in the Galton-Watson branching (GWB) system with branching rates $\{p_k, k \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. This is a reducible, homogeneous-discrete-time Markov chain with a state space consisting of two classes: $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\{0\}$ is absorbing state, and $\mathcal{S} \subset \mathbb{N}$ is the class of possible essential communicating states. We assume throughout this paper that $0 < p_0 + p_1 < 1$ and $m := \sum_{k\in\mathcal{S}} kp_k < \infty$. The parameter m is the average number of direct descendants of one individual in one-step generation. We are interested in the subcritical and supercritical cases, which are assigned $m < 1$ and $m > 1$ respectively. Denoting q be an extinction probability of the system initiated by a single founder, we recall that it is smallest nonnegative root of the fixed-point equation $f(s) = s$ on the domain of $\{s : s \in [0, 1]\}$, where

$$f(s) = \sum_{k \in \mathcal{S}_0} p_k s^k$$

is the offspring generating function (GF). The extinction probability $q = 1$ in subcritical case, and $q < 1$ when the system is supercritical. So, the supercritical system survives with positive probability; see [1].

Put into consideration n-step transition probabilities

$$P_{ij}(n) := \mathsf{P}\left\{Z(n+k) = j \mid Z(k) = i\right\} \qquad \text{for any} \quad k \in \mathbb{N}_0.$$

A corresponding probability GF $\sum_{k \in \mathcal{S}_0} P_{ij}(n) s^k = \left[f_n(s)\right]^i$, where

$$f_n(s) := \sum_{k \in \mathcal{S}_0} \mathsf{p}_k(n) s^k,$$

therein $\mathsf{p}_k(n) := P_{1k}(n)$. Then $f_n(0) = \mathsf{p}_0(n)$ is a vanishing probability of the system initiated by a single founder. This probability tends monotonously to q as $n \to \infty$, i.e. $\lim_{n\to\infty} \mathsf{p}_0(n) = q$. Furthermore $f_n(s) \to q$ as $n \to \infty$ uniformly in $s \in [0, 1)$; see [1, Ch.I.].

Let $R_n(s) := q - f_n(s)$. Denoting $\mathcal{H} := \min\{n \in \mathbb{N} : Z(n) = 0\}$ be an extinction time of the system initiated by a single founder, we note that $Q(n) := R_n(0) = \mathsf{P}\left\{n < \mathcal{H} < \infty\right\}$ is the probability of that the system survives at the time n but degenerates eventually. For the subcritical case $\mathsf{P}\{\mathcal{H} < \infty\} = 1$ and, hence $Q(n) = \mathsf{P}\{Z(n) > 0\}$. In this case, Kolmogorov [2] proved that if

$$f''(1-) < \infty \quad \text{for} \quad m < 1 \qquad [\mathsf{K}]$$

then $Q(n)$ admits the following asymptotic representation:

$$Q(n) = \mathcal{K} m^n \left(1 + o(1)\right) \qquad \text{as} \quad n \to \infty, \tag{1}$$

where $\mathcal{K}$ is the well-known Kolmogorov constant, but it does not have an explicit form here.

This report aims to improve and generalize the asymptotic formula (1) to the noncritical case. Denote

$$\beta := f'(q) \qquad \text{and} \qquad \gamma_q := \frac{f''(q)}{2\beta\,(1-\beta)}.$$

Theorem 1. Let $m \neq 1$. If Kolmogorov condition [K] is satisfied,
i) then

$$\frac{\mathsf{P}\left\{n < \mathcal{H} < \infty\right\}}{\mathcal{K}_q \beta^n} = 1 - B_q \mathcal{K}_q \beta^n \left(1 + o(1)\right) \qquad as \quad n \to \infty, \tag{2}$$

ii) and

$$\mathsf{E}\left[Z(n) \mid n < \mathcal{H} < \infty\right] = \frac{q}{\mathcal{K}_q}\left(1 + B_q \mathcal{K}_q \beta^n \left(1 + o(1)\right)\right) \qquad as \quad n \to \infty, \tag{3}$$

where

$$\mathcal{K}_q = \frac{q}{1 + q\gamma_q} \qquad \text{and} \qquad B_q = \frac{f''(q)}{2\beta}.$$

Remark. The principal novelties of Theorem 1 are as follows. First, it generalizes and asymptotically refine Kolmogorov's result (1) and the analogous theorem, established by Sevastyanov [3] only for the subcritical case. Secondly, the main terms on the right-hand side of both statements (2) and (3) involve an explicit form of the constant $\mathcal{K}_q$. Finally, the decrease rates decay of the second term in these asymptotic expansions are found.

Theorem 2. Let $m \neq 1$ and the condition $[\mathsf{K}]$ is satisfied. Then the following asymptotic relation holds:

$$\frac{\mathsf{p}_1(n)}{\beta^n} = \frac{1}{q^2}\mathcal{K}_q^2 \cdot \Big(1 + 2\gamma_q \mathcal{K}_q \beta^n \big(1 + o(1)\big)\Big) \qquad as \quad n \to \infty,$$

where $\mathcal{K}_q$ is the Kolmogorov constant appearing in Theorem 1.

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Periodic branching processes with immigration and their implicit multi-type representation

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Nowadays, there is a growing interest in non-Gaussian time series, particularly in series comprised of non-negative integers or counts, see the latest survey [1]. Count series arise in fields, such as agriculture, economics, epidemiology, finance, geology, meteorology, and sports. Many natural or human phenomena exhibit periodic behavior. A potentially promising model to describe periodic count time series is the periodic non-negative integer-valued autoregressive and moving average (PINARMA) model based on the binomial thinning operation, see [2]. A particular periodic and seasonal example of the PINARMA model is considered in [3].

In this talk, the generalized PINARMA model with period $S \in \mathbb{N}$, autoregressive orders $(p_s) \in \mathbb{N}_0^S$ and moving average orders $(q_s) \in \mathbb{N}_0^S$ is considered which is defined as

$$Y_{kS+s} = \sum_{i=1}^{p_s} \alpha_{k,s,i} \circ Y_{kS+s-i} + \sum_{j=1}^{q_s} \beta_{k,s,j} \circ \varepsilon_{kS+s-j} + \varepsilon_{kS+s},$$

$k \in \mathbb{Z}$, $s = 1, \ldots, S$, where $\{\alpha_{k,s,i}\circ \mid k \in \mathbb{Z}\}$ and $\{\beta_{k,s,j}\circ \mid k \in \mathbb{Z}\}$ are sequences of identically distributed generalized thinning operators with non-negative means $\alpha_{s,i}$, $i = 1, \ldots, p_s$ and $\beta_{s,j}$, $j = 1, \ldots, q_s$, $s = 1, \ldots, S$, respectively, and the immigration sequence $\{\varepsilon_{kS+s} \mid k \in \mathbb{Z}, s = 1, \ldots, S\}$ consists of $\mathbb{N}_0$-valued random variables. The parameters $\alpha_{s,i}$'s and $\beta_{s,j}$'s are called periodic autoregressive and moving average coefficients. We denote by Y_{kS+s} the series during the sth season of period k. We recall that the generalized thinning operator $\alpha\circ$ is defined as the random sum $\alpha \circ Y := \sum_{j=1}^{Y} \alpha_j$, where $\{\alpha_j \mid j \in \mathbb{N}\}$ is a sequence of independent identically distributed $\mathbb{N}_0$-valued random variables with non-negative finite mean α, and Y is a $\mathbb{N}_0$-valued random variable which is mutually independent of $\{\alpha_j\}$. All thinning operators $\alpha_{k,s,i}\circ$'s and $\beta_{k,s,j}\circ$'s are supposed to be mutually independent and independent of the immigration process $\{\varepsilon_t\}$. The generalized PINARMA model can be interpreted as a higher-order branching process with immigration in varying environment.

We show that the generalized PINARMA model can be written, with the help of the generalized matricial thinning operator, in a S-dimensional stationary vector model form, which is called the VINARMA model, as

$$\boldsymbol{Y}_k = \sum_{i=0}^{p} A_{k,i} \circ \boldsymbol{Y}_{k-i} + \sum_{j=0}^{q} B_{k,j} \circ \boldsymbol{\varepsilon}_{k-j},$$

where $\boldsymbol{Y}_k := (Y_{kS+S}, Y_{kS+S-1}, \ldots, Y_{kS+1})^\top$ and $\boldsymbol{\varepsilon}_k := (\varepsilon_{kS+S}, \varepsilon_{kS+S-1}, \ldots, \varepsilon_{kS+1})^\top$, $k \in \mathbb{Z}$, ($\top$ denotes transpose). In this equation, the entries of the matricial thinning operators $A_{k,i}\circ$'s and $B_{k,j}\circ$'s are defined with the help of the thinning operators $\alpha_{k,s,i}\circ$'s and $\beta_{k,s,j}\circ$'s, and the autoregressive and moving average orders are $p := [\max_{1\le s\le S}(p_s - s)/S] + 1$ and $q := [\max_{1\le s\le S}(q_s - s)/S] + 1$, respectively, where $[x]$ denotes the greatest integer less than or equal to a real x. The sequences $\{A_{k,i}\circ \mid k \in \mathbb{Z}\}$ and $\{B_{k,j}\circ \mid k \in \mathbb{Z}\}$ consist of identically distributed generalized matricial thinning operators with mean matrices A_i and B_j, respectively, for all $i = 0, 1, \ldots, p$, $j = 0, 1, \ldots, q$. A_i's and B_j's are non-negative square matrices of dimension S. In particular, A_0 is strictly upper triangular, and B_0 is upper triangular with unit diagonal. If $A_0 \neq 0$ then the VINARMA model is called proper implicit since $\boldsymbol{Y}_k$ appears on both sides of the equation. We suppose that $\{\boldsymbol{\varepsilon}_k\}$ is a sequence of independent identically distributed random vectors and $\mathsf{P}(\boldsymbol{\varepsilon} = 0) < 1$.

An essentially optimal and simple spectral criterion based on the model parameters is given for the existence and uniqueness of a solution to the above stochastic models. Let $\rho(M)$ denote the spectral radius of a matrix M. Let $\mathcal{G}_k$ denote the σ-algebra generated by the matricial thinning operators $A_{l,i}\circ$, $i = 0, 1, \ldots, p$, $B_{l,j}\circ$, $j = 0, 1, \ldots, q$, and r.v.'s $\boldsymbol{\varepsilon}_l$, $l \le k$, for all $k \in \mathbb{Z}$. $\{\boldsymbol{Y}_k\}$ is called a non-anticipative solution to the VINARMA model if the set of random variables $\{\boldsymbol{Y}_j \mid j \le k\}$ are mutually independent of the set of matricial thinning operators $\{A_{l,i}\circ, B_{l,j}\circ \mid l > k, i = 0, 1, \ldots, p, j = 0, 1, \ldots, q\}$ and the immigration vectors $\{\boldsymbol{\varepsilon}_j \mid j > k\}$ for all $k \in \mathbb{Z}$.

Theorem 1. Suppose that $\mathsf{E}\|\boldsymbol{\varepsilon}_0\| < \infty$ and the matrices $\{A_0, A_1, \ldots, A_p\}$ satisfy $\rho(A_0 + A_1 + \ldots + A_p) < 1$. Then the VINARMA model has a unique non-anticipative solution $\{\boldsymbol{Y}_k\}$ which can be expressed as the almost sure convergent infinite series

$$\boldsymbol{Y}_k = \sum_{j=1}^{\infty} \boldsymbol{Z}_k^{(j)},$$

$k \in \mathbb{Z}$, where $\boldsymbol{Z}_k^{(n)}$ denotes the number of nth generation offspring of immigrants at time

k. The $\mathbb{N}_0^S$-valued stochastic process $\{\boldsymbol{Y}_k\}$ is $\{\mathcal{G}_k\}$-adapted, ergodic and strictly stationary pth order homogeneous Markov chain with finite mean.

We provide a complete description of the probabilistic structure, among others, the mean and the covariance function of unique solutions to the PINARMA and VINARMA models. Two infinite series representations, moving average and immigrant generation, are also derived. A successive approximation procedure based on immigrant generation representation is proposed for constructing the unique solution to the models, which can be used to simulate the process efficiently, see [5]. The Yule-Walker method is applied to estimate the parameters of the PINARMA and VINARMA models. To investigate the asymptotic behavior of Yule-Walker estimators, we prove limit theorems extending some known results of periodic multi-type Galton-Watson branching process without immigration, see [4].

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On refinements of the asymptotic expansion of the continuation of critical branching processes

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Keywords: Critical branching processes, probability of the continuation.

For the probability of the continuation of critical branching processes, an asymptotic expansion is obtained under the assumption of the existence of factorial moments α_k for $k = 4, 5, ..., m, \quad m < \infty$.

Let Z_n, $n = 0, 1, 2, ...$, be a branching process with discrete time and one type of particles and $Q_n = 1 - P_0(n)$ be the probability of the continuation of the process.

The asymptotic behavior of probability Q_n for discrete time was studied by A. N. Kolmogorov [1]. The results of A. N. Kolmogorov for processes with continuous time were obtained by B. A. Sevastyanov [2].

A literature review on the issues of limit theorems and local limit theorems, and in particular, refinement of the asymptotic expansion for probability Q_n, is briefly presented in the publication by S. V. Nagaev and R. Mukhamedkhanova [3].

In this abstract, we consider some refinements of the theorems proven in [3] on the asymptotic expansion for probability Q_n.

The following theorems were proven in [3] for critical branching processes ($A = 1$), (see [3], pp. 96-97):

Theorem 1. If $A = 1, B > 0, C < \infty,$ then as $n \to \infty$:

$$Q_n = \frac{2}{Bn} + \left(\frac{4C}{3B^3} - \frac{2}{B}\right)\frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right) \tag{1}$$

Theorem 2. If $A = 1, B > 0, D < \infty,$ then as $n \to \infty$:

$$Q_n = \frac{2}{Bn} + \left(\frac{4C}{3B^3} - \frac{2}{B}\right)\frac{\ln n}{n^2} + \frac{4K}{B^2 n^2} + O\left(\frac{\ln n}{n^3}\right), \tag{2}$$

where K is some constant dependent on the form of $F(x)$.

In [3], the authors reported that their methods for proving relations (1)–(2) are suitable for the case when factorial moments of a higher order exist.

Although not significant, The authors also considered the case when there are factorial moments $\alpha_k = F^{(k)}(1) < \infty$ for $k \geq 4$.

Now we proceed to the consideration of the case when there are factorial moments $\alpha_k < \infty$, where $k = 4, 5, ..., m,\ \ m < \infty$.

Theorem 3. If $A = 1,\ B > 0,\ \alpha_k < \infty,\ k \geq 4$, then as $n \to \infty$ for Q_n the following asymptotic formula holds :

$$Q_n = \left\{\sum_{i=0}^{l}(-1)^i\left(\frac{2}{B}\right)^{2i+1}\frac{L_1^i \ln^i n}{n^{i+1}} + \left(\frac{2}{B}\right)^{2l+1}\frac{L_1^{l+1}\ln^{l+1} n}{n^{l+2}} + \right.$$
$$\left. + \left[\sum_{i=0}^{l}(-1)^{i+1}\left(\frac{2}{B}\right)^{2(i+1)}\frac{L_1^i \ln^i n}{n^{i+2}}\right](K_m + M_{nm})\right\}\left(1 + o\left(\frac{\ln n}{n}\right)\right) \tag{3}$$

where $m \geq 4$,

$$K_m = 1 + L_1\left[1 + \frac{2c_1}{B} + O\left(\frac{1}{n}\right)\right] + \sum_{j=2}^{m-2} L_j\left(1 + I_j\right) - \sum_{j=2}^{m-2} L_j \zeta\left(j\right),$$

$$M_{nm} = \sum_{j=2}^{m-2} L_j R_{nj},\ \ R_{nj} = \frac{2}{B}\sum_{t=n}^{\infty}\frac{1}{t^j}\left(1 + o\left(1\right)\right),\ \ L_1 = \frac{B^2}{4} - \frac{C}{6}$$

$$I_j = \sum_{t=1}^{\infty} Q_t^j,\ \ \zeta\left(j\right) = \sum_{t=1}^{\infty}\frac{1}{t^j} - Eyler\ zeta\ function,$$

and coefficients L_i, $i = 1, 2, ..., m-1$, depend only on factorial moments $\alpha_k < \infty$, $k = 2, 3, ..., m$, $c_1 = 0,577216...-$ is the Eyler's constant, and parameter $l = 1, 2, ...$, is defined below.

Comment

Parameter l in formula (3) determines the number of steps in the process of division with a remainder. From a practical point of view, it is more advantageous to assume that $l = 1$ or 2, or 3, etc. Obviously, parameter l allows us to determine the number of asymptotic terms in the asymptotic expansion for Q_n the probability of continuation of critical branching processes with discrete time.

It is easy to see that the expansion (3) contains $(l+1)(m-1)+1$, $(m \geq 4)$ of asymptotic terms, each of which has an explicitly defined constant coefficient, depending only on factorial moments $\alpha_k < \infty$, and the independent parameters l and m take the values of $l = 1, 2, ...$, $m = 4, 5, ...$.

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Explosion phenomena for Fleming-Viot-type processes

Martin Kolb

We will consider branching processes of Fleming-Viot-type which are driven by Bessel type diffusions on the non-negative reals with drift pointing towards the origin. This process has been previously considered by Burdzy et al., where the authors have been able to fully characterize, when the two particle system is well-defined. In the case of more than two particles a sufficient condition for the process to be well-defined was derived. In the talk we recall these results and present a new abstract critierium, which in the case of three particles sharpens the previous results significantly.

A limit theorem for the critical Galton-Watson branching processes

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Suppose that $\{\xi(k,j)\ ,\ k,j\in\mathbf{N}\}$ be a sequence of independent identically distributed random variables taking non-negative integer values. Let the random variable $\xi(1,1)$ have the distribution

$$p_k = P(\xi(1,1)=k)\,,\ \ k=0,\,1,...,$$

with the generating function

$$F(s):=Es^{\xi(1,1)}=\sum_{k=0}^{\infty}p_k s^k\,,\ 0\leq s\leq 1,$$

and $p_0+p_1\neq 1$. Consider the process $W(k)$, $k\geq 0$ defined by the following recurrent relation:

$$W(0)=\eta\ ,\ \ W(n)=\sum_{j=1}^{W(n-1)}\xi(n,j)\ ,\ n\in\mathbf{N}, \tag{1}$$

here η is a random variable that takes positive integer values and independent on the sequence of random variables
$\{\xi(k,j)\ ,\ k,j\in\mathbf{N}\}$.

We call the process $\{W(k)\ ,\ k\geq 0\}$ the Galton-Watson process starting with a random number of particles η. It is well known [1], the asymptotic state of the process $\{W(k)\ ,\ k\geq 0\}$ depends on the mean value of the random variable $\xi(1,1)$, and it is divided into the classes as follows. It is clear that $F'(1)=E\xi(1,1)$. The process (1) is called subcritical, critical and supercritical if $F'(1)<1$, $F'(1)=1$ and $F'(1)>1$, respectively.

In this work, we consider only critical processes.

We denote the Galton-Watson process generated by the i-th particle in the initial state by $W_i(n),\ n=0,1,...$. Obviously, $W_i(n),\ n=0,1,...\ ,\ i\geq 1$ form independent and identically distributed Galton-Watson branching processes. It is known [1] that $W(n)$ can be represented as

$$W(n)=\sum_{i=1}^{\eta}W_i(n)\,,\ n\in\mathbf{N}. \tag{2}$$

Independence of random variables η and $\xi(i,j)$, $i \geq 1$, $j \geq 1$ implies independence of $W_i(n)$ and the random variable η. Denote by $P(n)$ the probability of degeneration of the process $\{W(k)\ ,\ k \geq 0\}$ at the n-th step, i.e. $P(n) = P(W(n) = 0)$. We denote by $R(n)$ the probability of continuation of the process $W_1(n)$ at the n-th step, i.e. $R(n) = P(W_1(n) > 0)$. In what follows, we need the following designations:

$$Q(n) = 1 - P(n)\ ,\quad h(s) := Es^{\eta}\ ,\quad H_n(s) := Es^{W(n)}\ ,\ A = h'(1)\ ,\ \sigma^2 = F''(1)\ ,$$

$F_0(s) = s$, $F_1(s) = F(s)$, $F_n(s) = F(F_{n-1}(s))$ is the n-th iteration of $F(s)$.

Further the sign $a_n \sim b_n$ indicates that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$.

In 1968, Slack [3] considered the case of

$$F(s) = s + (1-s)^{1+\alpha} L(1-s)\ ,\ \alpha \in (0,1]\,, \tag{S}$$

here $L(x)$ is a slowly varying function on a neighborhood of zero, and obtained the following:

$$(1 - F_n(0))^{\alpha} L(1 - F_n(0)) \sim \frac{1}{\alpha n}, \tag{3}$$

$$\lim_{n\to\infty} E\left(\exp\left\{-\lambda(1 - F_n(0))W(n)\right\}/W(n) > 0\right) = 1 - \lambda\left(1 + \lambda^{\alpha}\right)^{-1/\alpha}\ ,\quad \lambda > 0. \tag{4}$$

This result implies the result by Yaglom [2] if $\alpha \equiv 1$ and $F''(1) < \infty$. It should be noted that in the case considered by Slack, the equality $F''(1) = \infty$ can be satisfied.

In [4], K.V. Mitov, G.K. Mitov, N.M. Yanev considered the critical case $(F'(1) = 1)$ when the second factorial moment was finite: $F''(1) = \sigma^2 < \infty$, and the generating function of the number of particles in the initial state was satisfied the condition

$$h(s) = 1 - (1-s)^{\theta} L_0\left(\frac{1}{1-s}\right)\ ,\ \theta \in (0,1), \tag{M}$$

here $L_0(x)$ is a slowly varying function at infinity, and obtained the following results:

$$P(W(n) > 0) = 1 - h(F_n(0)) \sim (\sigma^2 n)^{-\theta} L_0(n), \tag{5}$$

$$\lim_{n\to\infty} E\left(\exp\left\{-\lambda(1 - F_n(0))W(n)\right\}/W(n) > 0\right) = 1 - \lambda^{\theta}(1+\lambda)^{-\theta}\ ,\ \lambda > 0. \tag{6}$$

With the help of Tauber's theorems, it is not difficult to see that condition (M) implies that the average number of particles in the initial state is infinitely. But it follows from (7) that in this case, too, the critical Galton-Watson process will degenerate with probability 1.

We got the following result.

Theorem. If the conditions (M) and (S) are satisfied, then

$$\lim_{n\to\infty} E\left(\exp\left\{-\lambda(1 - F_n(0))W(n)\right\}/W(n) > 0\right) = 1 - \lambda^{\theta}(1+\lambda^{\alpha})^{-\theta/\alpha}\ ,\ \lambda > 0.$$

In the case of $F''(1) < \infty$, Theorem implies the result by Mitov, Mitov, and Yanev. If we set formal $\theta = 1$, $\alpha = 1$ in the last Laplace substitution, we get the Laplace substitution $(1+\lambda)^{-1}$ of the exponential distribution.

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A periodic branching random walk with immigration on Z^d

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Keywords: Branching random walks, immigration, periodic potential, direct integral decomposion, asymptotic behavior.

We consider a continuous-time branching random walk with immigration on Z^d with branching sources located periodically. We assume that the processes of branching and immigration are independent and the intensity matrices of random walk of immigration centers and particles are denoted by $\{a_i(v,u)\}_{v,u\in\mathsf{Z}^d}$, $i=0,1$ respectively. The intensity of branching and immigration at a point v are denoted by $\beta(v)$ and $\alpha(v)$ respectively.

Let A_i and Q be the operators $l_2(\mathsf{Z}^d)\to l_2(\mathsf{Z}^d)$ such that

$$(A_i f(\cdot))(v)=\sum_{u\in\mathsf{Z}^d} a_i(v,u)f(u),\quad i=0,1,$$

$$Qf(v)=\beta(v)f(v).$$

During the talk, the equation for the mean number of particles at the moment t at the point u with starting point v will be presented and the following theorem will be formulated:

Theorem 1. Let $\lambda_1(0)$ be the upper bound of the spectrum of the operator A_1+Q.

1. If $\lambda_1(0)>0$, then for every $u,v\in\mathsf{Z}^d$ there exists $C=C(u,v,d)$ such that

$$M(v,u,t)=C(v,u,d)\cdot\frac{e^{\lambda_1(0)t}}{t^{d/2}}\left(1+O\left(\frac{1}{t}\right)\right),\quad t\to\infty.$$

2. If $\lambda_1(0)<0$, then for every $u,v\in\mathsf{Z}^d$ there exists $\tilde{C}=\tilde{C}(u,v,d)$ such that

$$m_0(v,u,t)=\tilde{C}(v,u,d)\cdot\frac{1}{t^{d/2}}\left(1+O\left(\frac{1}{t}\right)\right),\quad t\to\infty.$$

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Asymptotic fluctuations of supercritical general branching processes

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The Crump-Mode-Jagers (CMJ) process is a fairly general branching process that unifies and extends earlier models of individual-based branching processes. Nerman's celebrated law of large numbers (1981) states that, for a supercritical CMJ process $(\mathcal{Z}_t)_{t\geq 0}$, under some mild assumptions, $e{-}\alpha t\mathcal{Z}_t$ converges almost surely as $t \to \infty$ to aW. Here, $\alpha > 0$ is the Malthusian parameter, a is a constant and W is the limit of Nerman's martingale, which is positive on the event that the population survives.

Time reversal of spinal processes for linear and non-linear branching processes near stationarity

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We consider a stochastic individual-based population model with competition, trait structure affecting reproduction and survival, and changing environment. The changes of traits are described by Gaussian or jump processes, and the dynamics can be approximated in large population by a non-linear PDE with a non-local mutation operator. Using the fact that this PDE admits a non-trivial stationary solution, we can approximate the non-linear stochastic population process by a linear birth-death process where the interactions are frozen, as long as the population remains close to this equilibrium. This allows us to derive, when the population is large, the equation satisfied by the ancestral lineage of an individual uniformly sampled at a fixed time T, which is the path constituted of the traits of the ancestors of this individual in past times before T.

This process is a time inhomogeneous Markov process, but we show that the time reversal of this process possesses a very simple structure (e.g. time-homogeneous and independent of T).

Bayesian inference in controlled branching processes via ABC methodology

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Keywords: approximate Bayesian computation, Bayesian inference, controlled branching processes, logistic growth population model.

In a Bayesian setting, we aim at estimating the posterior distribution of the parameters of interest in controlled branching processes without determining the exact likelihood function or prior knowledge of the maximum number of offspring that an individual can give birth.

To that end, we have developed approximate Bayesian computation (ABC) algorithms for branching processes. More precisely, to estimate the maximum number of children per individual we present an ABC rejection method for model choice based on the comparison with the observed raw data. In the next stage, using an appropriate summary statistic we approximate the posterior distributions of the target parameters by employing a tolerance-rejection method along with a post-sampling correction.

We illustrate the methodology by means of some examples developed with the statistical software R.

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The total progeny in the positive recurrent Q-processes

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Keywords: Branching system, Q-process, Markov chain, extinction time, total progeny, positive recurrent.

Let $\{Z(n), n \in \mathsf{N}_0\}$ GWB system with branching rates $\{p_k, k \in \mathsf{N}_0\}$, where $\mathsf{N} = \{1, 2, \ldots\}$, $\mathsf{N}_0 = \{0\} \cup \mathsf{N}$ and the variable $Z(n)$ denote the population size at the moment n in the system. The evolution of the system occurs according to the following mechanism. Each individual lives a unit length life time and then gives $k \in \mathsf{N}_0$ descendants with probability p_k. This process is a reducible, homogeneous-discrete-time Markov chain with a state space consisting of two classes: $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\{0\}$ is absorbing state, and $\mathcal{S} \subset \mathsf{N}$ is the class of possible essential communicating states. Throughout the paper assume that $p_0 > 0$ and $p_0 + p_1 > 0$ which called the Schröder case. We suppose that $p_0 + p_1 < 1$ and $m := \sum_{k\in\mathcal{S}} kp_k < \infty$.

Considering transition probabilities

$$P_{ij}(n) := \mathsf{P}\left\{Z(n+k) = j \;\middle|\; Z(k) = i\right\} \quad \text{for} \quad \text{any} \quad k \in \mathsf{N}_0$$

we observe that the corresponding probability generating function (GF)

$$\sum_{k\in\mathcal{S}_0} P_{ij}(n)s^k = \left[f_n(s)\right]^i, \tag{1}$$

where $f_n(s) := \sum_{k\in\mathcal{S}_0} \mathsf{p}_k(n)s^k$, therein $\mathsf{p}_k(n) := P_{1k}(n)$ and $f_n(s)$ is n-fold iteration of the offspring GF $f(s) := \sum_{k\in\mathcal{S}_0} p_k s^k$. Needless to say that $f_n(0) = \mathsf{p}_0(n)$ is a vanishing probability of the system initiated by one individual. Note that this probability tends as $n \to \infty$ monotonously to q, which called an extinction probability of the system, i.e. $\lim_{n\to\infty} \mathsf{p}_0(n) = q$. The extinction probability $q = 1$ if $m \le 1$ and $q < 1$ if $m > 1$. Based on this, according to the values of the parameter m, the system is called *sub-critical* if $m < 1$, *critical* if $m = 1$ and *super-critical* if $m > 1$.

Further we are dealing with the GWB system conditioned on the event $\{n < \mathcal{H} < \infty\}$, where

$$\mathcal{H} := \min\{n \in \mathsf{N} : Z(n) = 0\}$$

is the extinction time. Let $\mathsf{P}_i\{*\} := \mathsf{P}\left\{* \;\middle|\; Z(0) = i\right\}$ and define conditioned probability measure

$$\mathsf{P}_i^{\mathcal{H}(n+k)}\{*\} := \mathsf{P}_i\left\{* \;\middle|\; n+k < \mathcal{H} < \infty\right\} \quad \text{for} \quad \text{any} \quad k \in \mathsf{N}.$$

In [2] proved, that

$$\mathcal{Q}_{ij}(n) := \lim_{k\to\infty} \mathsf{P}_i^{\mathcal{H}(n+k)}\left\{Z(n) = j\right\} = \frac{jq^{j-i}}{i\beta^n} P_{ij}(n), \tag{2}$$

where $\beta := f'(q)$. Observe that $\sum_{j\in\mathsf{N}} \mathcal{Q}_{ij}(n) = 1$ for each $i \in \mathsf{N}$. Thus, the probability measure $\mathcal{Q}_{ij}(n)$ can determine a new population growth system with the state space $\mathcal{E} \subset \mathsf{N}$ which we denote by $\{W(n), n \in \mathsf{N}_0\}$. This is a discrete-homogeneous-time irreducible Markov chain defined in [2] and called *the Q-process*. Undoubtedly $W(0) \stackrel{d}{=} Z(0)$ and transition probabilities

$$\mathcal{Q}_{ij}(n) := \mathsf{P}\left\{W(n) = j \;\middle|\; W(0) = i\right\} = \mathsf{P}_i\left\{Z(n) = j \;\middle|\; \mathcal{H} = \infty\right\},$$

so that the Q-process can be interpreted as a "long-living" GWB system. Put into consideration a GF $w_n^{(i)}(s) := \sum_{j\in\mathcal{E}} \mathcal{Q}_{ij}(n)s^j$. Then from (1) and (2) we obtain

$$w_n^{(i)}(s) = \left[\frac{f_n(qs)}{q}\right]^{i-1} \cdot w_n(s), \tag{3}$$

where the GF $w_n(s) := w_n^{(1)}(s) = \mathsf{E}\left[s^{W(n)} \;\middle|\; W(0) = 1\right]$ has a form of $w_n(s) = sf_n'(qs)/\beta^n$ for all $n \in \mathsf{N}$. Using iterations for $f(s)$ in (3) leads to the following functional equation:

$$w_{n+1}^{(i)}(s) = \frac{w(s)}{f_q(s)} w_n^{(i)}\big(f_q(s)\big), \tag{4}$$

where $w(s) := w_1(s)$ and $f_q(s) = f(qs)/q$. Thus, Q-process is completely defined by setting the GF

$$w(s) = s\frac{f'(qs)}{\beta}. \tag{5}$$

An evolution of the Q-process is in essentially regulated by the structural parameter $\beta > 0$. In fact, as it has been shown in [2], that $\mathcal{E}$ is positive recurrent if $\beta < 1$ and $\mathcal{E}$ is transient if $\beta = 1$. On the other hand, it is easy to be convinced that positive recurrent case $\beta < 1$ of Q-process corresponds to the non-critical case $m \neq 1$ of the initial GWB system. Note that $\beta \leq 1$ and nothing but.

In this paper, we deal with the positive recurrent case assuming that first moment $\alpha := w'(1-)$ be finite. Then differentiating (5) on the point $s = 1$ we obtain $\alpha = 1 + \gamma_q (1 - \beta)$, where $\gamma_q := qf''(q)/\beta(1-\beta)$. It follows from (3) that $\mathsf{E}_i W(n) := \mathsf{E}\left[W(n) \,\Big|\, W(0) = i\right] = (i-1)\beta^n + \mathsf{E}W(n)$, where $\mathsf{E}W(n) = 1 + \gamma_q\big(1 - \beta^n\big)$.

It is obvious, that when initial GWB system is sub-critical, then the condition $\alpha < \infty$ is this is equivalent to that $f''(1-) < \infty$. Further we everywhere will be accompanied by this condition by default. Our purpose is to investigate asymptotic properties of a random variable

$$S_n = W(0) + W(1) + \ldots + W(n-1),$$

denoting the total number of individuals that have existed up to the n-th generation in Q-process. By analogy with branching systems, this variable is of great interest in studying the deep properties of the Q-process. Our main results are analogues of central limit theorem and law of large numbers for S_n.

Theorem 1. Let $\beta < 1$ and $\alpha < \infty$. Then

$$\frac{S_n - \mathsf{E}S_n}{\mathcal{K}_n} \to \mathcal{N}_{0,\sigma^2} \quad \text{as} \quad n \to \infty,$$

where $\mathcal{N}_{0,\sigma^2}$ – a normal distribution function with zero mean and finite variance of $\sigma^2 > 0$ and $\mathcal{K}_n := \mathcal{O}^*(\sqrt{n})$.

Theorem 2. Let $\beta < 1$ and $\alpha < \infty$. Then there exists slowly varying function at infinity $L(*)$ such that

$$\left|\mathsf{P}\left\{\frac{S_n - \mathsf{E}S_n}{\mathcal{K}_n} < x\right\} - \mathcal{N}_{0,\sigma^2}\right| \leq \frac{L(n)}{n^{1/4}}$$

uniformly in x.

Theorem 3. Let $\beta < 1$ and $\alpha < \infty$. Then the distribution of S_n/n converges weakly to the degenerate distribution concentrated at the point $1 + \gamma_q$, i.e.

$$\mathsf{P}\left\{\frac{S_n}{n} < x\right\} \Rightarrow I_{1+\gamma_q}(x) \qquad \text{and} \qquad I_{1+\gamma_q}(x) = \begin{cases} 0, & \text{if} \quad x \leq 1 + \gamma_q, \\ 1, & \text{if} \quad x > 1 + \gamma_q. \end{cases}$$

Moreover there exists slowly varying function at infinity $L_\gamma(*)$ such that

$$\left|\mathsf{P}\left\{\frac{S_n}{n} < x\right\} - I_{1+\gamma_q}(x)\right| \leq \frac{L_\gamma(n)}{\sqrt{n}}$$

uniformly in x.

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Gaussian waves in BBM with mean-dependent branching

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We consider a continuous-space analogue of a population model introduced by Yu, Etheridge and Cuthbertson. We prove a hydrodynamic limit result that allows us to show that for a large total population size, at large times the empirical distribution of the particle positions evolves approximately according to an accelerating Gaussian wave. Based on joint work with Erin Beckman.

Functional limit theorems and the asymptotic normality of estimators based on partial observations

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Keywords: Branching process; immigration; restricted observation; offspring mean.

The asymptotic normality of estimators of the offspring mean is important for applications. In the case when the population sizes are partially observed the limit theorems for the process are not sufficient to obtain asymptotic distributions for known estimators. As a result the asymptotic normality of the estimators based on partial observations has not been obtained in its standard form and leads to consider various modifications of the estimators. In the talk, we demonstrate that the functional limit theorems for the critical partially observed process with generation-dependent immigration allow to show that the original (non-modified) estimators are asymptotically normal. For this we first extend known functional limit theorems for fully observed processes obtained by Wei and Winnicki [1] and Rahimov [2], respectively, to the case of partial observations which is of independent interest as well. Then, we use the new theorems to obtain desired asymptotic normality.

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Branching process for the solution of semi-linear Helmholtz boundary value problem

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Keywords: Helmholtz equation, Dirichlet problem, Monte Carlo method, branching random process, walk on spheres, martingale, unbiased estimator.

As is known, there are many applied problems, the solution of which is associated with boundary value problems for nonlinear elliptic equations containing hyperbolic functions of unknown functions. For example, the equation $\Delta u(x) = k^2 \cdot ch(u)$ arises in problems of constructing surfaces of constant mean curvature in the hyperbolic space H^3 [1]. The equations $\Delta u(x) = k^2 \cdot sh(u)$ arise when solving problems of bio-molecular electrostatic theory [2]. This article considers a probabilistic approach to solving the first boundary value problem for the following two below equations. Let D be a bounded domain in R^3 with regular boundary Γ, $\varphi(x) \in C(\bar{\Gamma})$, $\psi(x) \in C(\bar{\Gamma})$. Consider the following Dirichlet problems:

$$-\Delta u(x) + c \cdot u(x) = g \cdot sh(u), \quad x \in D, \quad u|_\Gamma = \varphi;$$
$$-\Delta u(x) + c \cdot u(x) = g \cdot ch(u), \quad x \in D, \quad u|_\Gamma = \psi.$$

It is assumed that the functions $\varphi(x), \psi(x)$ and the coefficients c, g are such that there exists a unique continuous solution to these semilinear problems ([3], [4]). Assuming the existence of a solution for the problems, an unbiased estimator is constructed on the trajectories of the branching process walk on spheres. Unbiased and biased estimators for the solution of boundary value problems for the linear Helmholtz equation $\Delta u - cu = -g(x)$ were considered for $c(x) = const$ by G.A. Mikhailov and B.S. Elepov in [5], [6], for the variable case $c(x)$ in [7], [8] for the Dirichlet problem, N.A. Simonov in [9] for the mixed problem and the problem Neumann, A.S. Sipin in [10] for the Dirichlet problem for the equation

$$\Delta u + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + au = -g.$$

In the papers [11], [12] by G.A. Mikhailov and R.N. Makarov, estimates for the solution of boundary value problems for the linear Helmholtz equation are built on the basis of a special integral-difference equation using the process of walking on spheres with reflection from the border. In the works [13], [14] the Monte Carlo solution of one applied problem of biomolecular electrostatic theory for the linearized equation $\Delta u(x) = k^2 \cdot u$ was considered. The Monte Carlo solution of the Dirichlet problem for a nonlinear equation of the form

$$\Delta u(x) = \sum_{i=1}^{n} a_i(x) u^{2i}(x) + a_0(x)$$

was proposed by A.S.Rasulov in his work [15], [16]. G.A. Mikhailov in [7] studied special case the equation $\Delta u + u^n = 0$ and in [17], [18] G.M.Raimova applied for the equation

$$\Delta u(x) + c\, u(x) = \sum_{i=0}^{\infty} a_i(x) u^i(x).$$

In these work we will study a probabilistic representation of the solution of the Helmholtz boundary problem for the non-linear problem

$$-\Delta u(x) + cu(x) = g \cdot f(u), \quad x \in D, \quad u|_\Gamma = \psi$$

where, $f(u)$ in our case could be hyperbolic functions $sh(u)$ or $ch(u)$. Under the assumption of the existence of a solution, an unbiased estimator is constructed on the trajectories of the proposed branching process "walk on spheres". To do this, using Green's formula, a special integral equation is written that connects the value of the function with its integrals over a ball and a sphere of maximum radius centered at a point and entirely contained in the region under consideration. It is proved that under certain conditions there exists a fixed point for the nonlinear integral operator corresponding to the integral equation. In this case, the iteration process method converges and classical Monte Carlo methods could be used. A probabilistic representation of the solution of the problem in the form of the mathematical expectation of some random variable is obtained. In accordance with the probabilistic representation, a branching process of walk on spheres is constructed and an unbiased estimator of the solution of the problem with finite variance is constructed on its trajectories.

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On random walks in random environment with random local constraints

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Keywords: conditioned random walk, bounded local times, regenerative sequence, potential regeneration, separating levels, skip-free distributions, accompanying process.

Consider a d-dimensional random walk

$$S_t = (S_t[1], \dots, S_t[d]) = S_0 + \sum_{j=1}^{t} \xi_j, \quad t = 0, 1, 2, \dots,$$

on the integer lattice $\mathbb{Z}^d$, where $\xi_j = (\xi_j[1], \dots, \xi_j[d]) \in \mathbb{Z}^d$, $j = 1, 2, \dots$, are i.i.d. random vectors that do not depend on the initial value $S_0 \in \mathbb{Z}^d$. We assume that at any time $t = 0, 1, 2, \dots$ the number of possible/allowed visits to each state $x \in \mathbb{Z}^d$ is limited above by a counting number $H_t(x) \geq 0$. Let

$$T_* = \inf\{t \geq 0 : H_t(S_t) = 0\} \leq \infty$$

be the first time when the walk visits a state with zero number of possible/allowed visits to it. If T_* is finite, we assume that the random walk “freezes” at the time instant T_* (or it “dies”, or “is killed” at time T_*).

We assume also that, at any time $t < T_*$, the random walk jumps from S_{t-1} to S_t and changes the environment at point S_{t-1} by decreasing the number of remaining allowed visits by 1, so that

$$H_t(x) := H_{t-1}(x) - \mathbf{1}\{S_{t-1} = x\} \quad \text{for each} \quad x \in \mathbb{Z}^d \quad \text{and all} \quad 0 \le t \le T_*.$$

Thus, we consider a multidimensional integer-valued random walk in a changing random environment.

As a natural example, consider a model of a random walk on atoms of a "harmonic crystal". An electron jumps from one atom to another, taking from a visited atom for the next jump a fixed unit of energy, that cannot be recovered. Thus, if S_t is a position of the electron at time t, then it takes a unit of energy to make the next jump to position $S_{t+1} = S_t + \xi_{t+1}$, which may be in any direction from S_t since the ξ's are signed random variables. When the electron arrives at an atom with insufficient energy level, it "freezes" there.

We interpret the first coordinate $S_t[1]$ of S_t as its *height* and assume further that the height cannot increase by more than one unit:

$$\xi_t[1] \le 1 \quad a.s., \qquad t = 1, 2, \dots. \tag{1}$$

Under the skip-free property (1) we may define the hitting time $\alpha(n)$ of the level n:

$$\alpha(n) := \inf\{t \ge 0 : S_t[1] \ge n\} = \inf\{t \ge 0 : S_t[1] = n\}.$$

Our simplest result is that, under natural technical assumptions (see detailes in [1] or [2]), there exist positive constants $0 < q_\infty \le 1$ and $0 < c_0 < \infty$ such that we have the following relation:

$$\mathbf{P}(B_n) \sim c_0 q_\infty^n \quad \text{as} \quad n \to \infty, \qquad \text{where} \quad B_n := \{\alpha(n) < T_*\}. \tag{2}$$

Thus in (2) we have found an exact asymptotic for the probability of the event B_n that our random walk reaches the level n before it "freezes".

Secondly, we prove convergence of the conditional distributions:

$$\mathbf{P}((S_0, \dots, S_K) \in A \mid B_n) \to \mathbf{P}((\overline{S}_0, \dots, \overline{S}_K) \in A), \quad \text{as} \quad n \to \infty, \tag{3}$$

for any $k = 0, 1, 2, \dots$ and all $A \subset \mathbb{Z}^{(K+1)\times d}$, where $\mathbb{Z}^{(K+1)\times d}$ denotes the space of vectors $\vec{x} = (x_0, x_1, \dots, x_K)$ having d-dimensional vectors as their components. Further, we find that the limiting sequence $\{\overline{S}_k\}$ in (3) has a regenerative structure with an infinite sequence of random regenerative levels $\{\overline{\nu}_i\}$ and increases to infinity with a linear speed, i.e.

$$\overline{S}_n[1]/n \to a_1 \in [0, 1] \quad \text{a.s.} \qquad \text{as} \quad n \to \infty. \tag{4}$$

Our proofs of results (2) – (4) in [1] and [2] are based on establishing a number of representations for the distribution of the random walk $\{S_t\}$ in random environment $\{H_t(x)\}$, that is linked to the distribution of the limiting sequence $\{\overline{S}_t\}$. For example, it is shown in [2] that, under some technical assumptions,

$$\mathbf{P}(B_n) = \psi_0 q_\infty^n \mathbf{P}(\overline{B}_n), \quad \text{where} \quad \overline{B}_n := \cup_{m=0}^n \{\overline{\nu}_m = n\}, \tag{5}$$

for a well-defined positive constant ψ_0 and for q_∞ as in (2); and that

$$\mathbf{P}((S_0, \dots, S_K) \in A \mid B_n) = \mathbf{P}((\overline{S}_0, \dots, \overline{S}_K) \in A \mid \overline{B}_n), \tag{6}$$

for any $n \ge K = 0, 1, 2, \dots$ and all $A \in \mathbb{Z}^{(K+1)\times d}$. Here event $\overline{B}_n$ occurs if (and only if) the given number n is one of the regenerative levels of the limiting random walk.

In paper [3], we have found that the limiting process is not the only one for which such representations do exist. We showed that there exist random sequences $\{\overline{S}_t\}$, that are called "accompanying" sequences and that depend on the used below numbers n and q, and are such that

$$\mathbf{P}(B_n) = \psi_n(q) q^n \mathbf{P}(\overline{B}_n) \qquad \text{for all} \quad q \geq q_n > 0, \tag{7}$$

where $0 < q_n \leq 1$ and $0 < \psi_n(q) < \infty$ are well-defined constants. We have to underline that representation (7) may take place also in cases then formulas (2) – (6) do not hold because the number q_∞ does not exist.

Several generalizations of results from [2] and [3] will be presented in the talk.

Earliar, in [4], convergences (2) – (4) were proved in a particular case when

$$\mathbf{P}(\xi_1 = 1) = 1/2 = \mathbf{P}(\xi_1 = -1), \quad S_0 = 0 \quad \text{and} \quad H_0(x) = L_0 = \mathit{const} \geq 2$$

for all $x \in \mathbb{Z}$. The latter means that initially each atom has a fixed (the same for all) amount of energy L_0. Paper [4] has motivated us to introduce and study the generalized model.

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Pushed and pulled waves in population genetics

Emmanuel Schertzer

This talk is motivated by the stochastic F-KPP equation with Allee Effect

$$\partial_t u \;=\; \frac{1}{2}\partial_{xx} u + u(1-u)(1+Bu) + \sqrt{\frac{1}{N} u(1-u)}\eta$$

where η is a space-time white noise. Numerical results and heuristics by Physicists suggest the existence of an interesting phase transition between a pulled, a semi pushed and a fully pushed regime. First, I will start with a brief explanation of the three regimes and their biological implications. I will then introduce a toy model which mimics the qualitative behavior of the aforementioned model. This is a class of critical branching Brownian motions with inhomogeneous branching rates which can be treated analytically using recent methods which are interesting on their own (moments of random trees, k-spines). This is joint work with J. Tourniaire (University of Vienna/ISTA) and Felix Foutel–Rodier (Oxford).

Asymptotics for the site frequency spectrum associated with the genealogy of a birth and death process

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Keywords: Birth and death process, Coalescent point process, Site frequency spectrum.

Consider a birth and death process started from one individual in which each individual gives birth at rate λ and dies at rate μ, so that the population size grows at rate $r = \lambda - \mu$. Lambert [1] and Harris, Johnston, and Roberts [2] came up with methods for constructing the exact genealogy of a sample of size n taken from this population at time T. We use the construction of Lambert, which is based on the coalescent point process, to obtain asymptotic results for the site frequency spectrum associated with this sample. In the supercritical case $r > 0$, our results extend results of Durrett [3] for exponentially growing populations. In the critical case $r = 0$, our results parallel those that Dahmer and Kersting [4] obtained for Kingman's coalescent.

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Large Deviations of Bisexual Brancing Processes in Random Environment

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Keywords: Bisexual Branching Process in Random Environment, Processes with Immigration, Large Deviations, Cramer Condition

Bisexual branching processes (BBP) were introduced by D. Daley in [1]. He considered i.i.d. random vectors $(U_{i,j}, V_{i,j})$, $i, j \in \mathbb{N}$, with $\mathbb{N}_0 \times \mathbb{N}_0$ values, were $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ and defined BBP N_n by the equation

$$N_{n+1} = \min\left(\sum_{i=1}^{N_n} U_{n,i}, \sum_{i=1}^{N_n} V_{n,i}\right), \; n \geq 0, \quad N_0 = 1.$$

It corresponds to the simple probabilistic model – we have particles of two sexes, they form pairs ('mating units'), every pair produce a random vector (X, Y) of descendants. This model is called BBP with completely promiscuous mating. In general situation

$$N_{n+1} = L\left(\sum_{i=1}^{N_n} U_{n,i}, \sum_{i=1}^{N_n} V_{n,i}\right),\ n \geq 0,$$

were $L : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ is some function. The function L is called the mating function of the process.

This model was studied by D. Daley, D. Hull, F. Bruss, J. Bagley, M. Gonzalez, M. Molina. A review of some important results can be found in [2].

We consider bisexual branching processes in random environment (BBPRE) introduced by Ma in [3]. We suppose that there exists a infinite sequence $\boldsymbol{\eta}$ of i.i.d. r.v. η_i (random environment), consider mating function $L : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{R} \to \mathbb{N}_0$ and a family of two-dimensional $\mathbb{N}_0 \times \mathbb{N}_0$ distributions $\{L_z,\ z \in \mathbb{R}\}$. We assume that for given $\boldsymbol{\eta}$

- the mating function in the i-th generation is $L(\cdot, \cdot, \eta_i)$;
- the numbers of descendants $(U_{i,j}, V_{i,j})$, $j \in \mathbb{N}$ of the mating units of i-th generation are i.i.d. random vectors with the distribution L_{η_i}.

For this model in [3] the extinction was studied. We study large deviation probabilities in this model.

We introduce the condition

$$|L(x, y, z) - g(x, y, z)| \leq c_1(|x| + |y|)^{1-\delta}$$

for some Lipschitz (with respect to x, y) function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, satisfying $g(cx, cy, z) = cg(x, y, z)$ for all $x, y, c \in \mathbb{R}^+$, $z \in \mathbb{R}$. All popular mating functions L satisfy this condition. Under this condition we introduce

$$\chi_i = \ln \mathbf{E}_{\eta_i} U_{i,1}, \quad \zeta_i = \ln \mathbf{E}_{\eta_i} V_{i,1}, \quad \xi_i = g(\chi_i, \zeta_i, \eta_i).$$

We call the random walk $S_n = \xi_1 + \cdots + \xi_n$ the associated random walk for BBPRE N_n. Under Cramer conditions on ξ_i and some moment conditions on U_i, V_i we prove that

$$\mathbf{P}(\ln N_n \in [x, x + \Delta)) \sim I(x/n)\mathbf{P}(S_n \in [x, x + \Delta)),$$

were I is some function, $\Delta > 0$ is some constant, $x/n \in [\theta_1, \theta_2] \subseteq (m^*, m^+)$, where m^*, m^+ are some constants.

We also consider BBPRE with immigration and obtain similar results in this model.

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Coalescence in Bisexual Branching Processes

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Keywords: Coalescence, Branching Process, Bisexual, Poisson, Mating function.

The study of branching processes has long been recognized as a powerful tool in understanding the dynamics of populations and the evolution of species. Bisexual branching processes provide a valuable framework for investigating scenarios where reproduction is possible between individuals of different genders. One fundamental aspect of such processes is coalescence, which refers to the merging of ancestral lineages over time. This study delves into the mathematical modeling of these processes, exploring the dynamics of lineages across generations and the probability of coalescence events. Coalescence times, which signify the duration it takes for a pair of lineages to merge, have implications for understanding the genetic diversity and effective population size of species. Moreover, the concept of coalescence provides essential tools for studying population genetics and phylogenetics. In the literature, a lot of studies have focused on bisexual branching process. Also, a lot of studies have focused on coalescence problem in several variants of discrete time Galton Watson Branching Process. However, very few studies have explored the coalescence problem in Bisexual Branching Process. We consider a discrete time bisexual branching process and consider a few special cases of superadditive mating functions to obtain interesting theoretical results. Further, using extensive simulation, we also observe the phenomenon of coalescence by generalizing the process to multitype bisexual branching process, where there can be multiple types of individuals. Some interesting insights have been obtained using simulation.

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On the rate of convergence in limit theorems for fluctuation critical branching processes with immigration

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Keywords: Branching process; generating function; weak convergence.

Let $\{\xi_{k,j}, k, j \geq 1\}$ and $\{\varepsilon_k, k \geq 1\}$ be two independent collections of independent, random variables taking non-negative integer values such that $\{\xi_{k,j}, k, j \geq 1\}$ identical distributed. Let the sequence of random variables $X_k, k \geq 0$ be defined by the following recursive relations:

$$X_0 = 0, \quad X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \quad k = 1, 2, ... \tag{1}$$

Stochastic processes defined in this way often appear in population theory (see, for example, [1]) and are called branching processes with immigration.

Let $m = \mathrm{E}\xi_{1,1} < \infty$. The branching process (1) is called subcritical, critical and supercritical if $m < 1$, $m = 1$ and $m > 1$ respectively. We obtain estimates a for the rate of convergence of the distribution of fluctuations of a branching process with immigration to the normal law and the obtained estimates are applied to the case when the immigration flow $\{\varepsilon_k, k \geq 1\}$ is inhomogeneous and the mean value and variance of ε_n are regular and increasing.

Let us introduce the following notation: $m = \mathrm{E}\xi_{1,1}$, $\sigma^2 = D\xi_{1,1}$, $\gamma = \mathrm{E}\left|\xi_{1,1} - 1\right|^3 < \infty$, $\lambda_k = \mathrm{E}\varepsilon_k$, $b_k^2 = var\varepsilon_k$, $\theta_k = \mathrm{E}\left|\varepsilon_k - \lambda_k\right|^3$, $\Gamma_n = \sum\limits_{k=1}^{n} (\varepsilon_k - \lambda_k)$, $\mathrm{T}_n^2 = D\Gamma_n$, $\mathrm{H}_n^2 = \sigma^2 \sum\limits_{k=1}^{n} X_{k-1}$.

$A_n = \sum\limits_{k=1}^{n} \lambda_k$, $B_n^2 -$ some sequence of positive numbers, $\Phi_\sigma(x) -$normal distribution with mean zero and variance σ^2, $\Phi(x) -$standard normal distribution,

$$\Delta_n = \sup_{-\infty < x < \infty} \left| \mathrm{P}\left(\frac{X_n - A_n}{B_n} < x \right) - \Phi(x) \right|.$$

Let us agree to denote by $C, C_1, C_2, ... -$ positive absolute constants.

Theorem 1. Let $m = 1$, $\gamma < \infty$, $b_k^2 < \infty$, $k \in N$. Then the following inequality is true

$$\Delta_n \leq C_1 \left[\frac{\gamma}{\sigma^3}\right]^{1/4} \left[\frac{\sum\limits_{k=1}^{n} EX_{k-1}^{3/2}}{\left(\sum\limits_{k=1}^{n} A_{k-1}\right)^{3/2}} \right]^{1/4} +$$

$$+C_2\left[E\left|\frac{H_n^2}{D_n^2}-1\right|^{3/2}\right]^{1/4}+3\left(\frac{\mathrm{T}_n^2}{2\pi B_n^2}\right)^{1/3}+\frac{1}{\sqrt{2\pi e}}\left|\frac{B_n}{D_n}-1\right|.$$

Theorem 2. Let $m=1$, $0<\sigma^2<\infty$, random variables ε_k, $k\geq 1$ are independent and $\theta_k<\infty$, $k\geq 1$. Then the following inequality is true

$$\Delta_n\leq C\frac{1}{\mathrm{T}_n^3}\sum_{k=1}^{n}\theta_k+3\left(\frac{\sigma^2\sum\limits_{k=1}^{n}A_{k-1}}{2\pi B_n^2}\right)^{1/3}+\frac{1}{\sqrt{2\pi e}}\left(\frac{B_n^2}{\mathrm{T}_n^2}-1\right).$$

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Random walks conditioned to stay nonnegative and branching processes in nonfavorable random environment

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Keywords: Random walk, branching processes, conditional limit theorems.

Let $\{S_n,n\geq 0\}$ be a random walk whose increments belong without centering to the domain of attraction of an α-stable law $\{Y_t,t\geq 0\}$, i.e. $S_{nt}/a_n\Rightarrow Y_t,t\geq 0$, for some scaling constants a_n. Assuming that $S_0=o(a_n)$ and $S_n\leq\varphi(n)=o(a_n)\to\infty$, we prove several conditional limit theorems for the distribution of S_{n-m} given $m=o(n)$ and $\min_{0\leq k\leq n}S_k\geq 0$. These theorems complement the statements established by F. Caravenna and L. Chaumont [1].

Let, further, $\{Z(k),k\geq 0\}$ be a critical branching process evolving in random environment and $\{S_k,k\geq 0\}$ be its associated random walk. Using the results obtained for random walks, we continue the study of processes evolving in unfavorable environment initiated at [2] and [3] and investigate the distribution of the properly scaled process $\{\log Z_k,k\geq 0\}$ for the moments $k=n-m$ given that $m=o(n)$, $Z_n>0$ and $S_n\leq\varphi(n)=o(a_n)\to\infty$.

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Times of a branching process with immigration in varying environment attaining a fixed level

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Keywords: Branching processes, varying environments, immigration, regeneration

Consider a branching process $\{Z_n\}_{n\geq 0}$ with immigration in varying environment. For $a \in \{0, 1, 2, ...\}$, let $C = \{n \geq 0 : Z_n = a\}$ be the collection of times at which the population size of the process attains level a. We give a criterion to determine whether the set C is finite or not. Especially, if $a = 0$, C is just the set of regeneration times. For critical Galton-Watson process, we show that $|C \cap [0, n]|/\log n \to S$ in distribution, where S is an exponentially distributed random variable with $P(S > t) = e^{-t},\ t > 0$.

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A Countable-Type Branching Process Model for the Tug-of-War Cancer Cell Dynamics

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Keywords: Multitype branching process, cancer dynamics, negative selection, deleterious passenger mutations, Tug-of-War.

We consider a time-continuous Markov branching process of proliferating cells with a countable collection of types. Among-type transitions are inspired by the Tug-of-War process introduced by [1] as a mathematical model for competition of advantageous driver mutations and deleterious passenger mutations in cancer cells. We introduce a version of the model in which a driver mutation pushes the type of the cell L-units up, while a passenger mutation pulls it 1-unit down. The distribution of time to divisions depends on the type (fitness) of cell, which is an integer. The extinction probability given any initial cell type is strictly less than 1, which allows us to investigate the transition between types (type transition) in an infinitely long cell lineage of cells. The analysis leads to the result that under driver dominance, the type transition process escapes to infinity, while under passenger dominance, it leads to a limit distribution. Implications in cancer cell dynamics and population genetics are discussed.

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Near-critical branching processes considered as Markov chains with small drift

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Let Z_n be a state-dependent branching process with migration. In the talk I shall discuss an approach to this rather wide class of processes, which is based on transformation $X_n = \sqrt{Z_n}$ and which allows one to obtain the results for branching processes without using generating functions.

Growth-fragmentation and quasi-stationary methods

Alex Watson

Abstract: A growth-fragmentation is a stochastic process representing cells with continuously growing mass and sudden fragmentation. Growth-fragmentations are used to model cell division and protein polymerisation in biophysics. A topic of wide interest is whether or not these models settle into an equilibrium, in which the number of cells is growing exponentially and the distribution of cell sizes approaches some fixed asymptotic profile. In this work, we present a new spine-based approach to this question, in which a cell lineage is singled out according to a suitable selection of offspring at each generation, with death of the spine occurring at size-dependent rate. The quasi-stationary behaviour of this spine process translates to the equilibrium behaviour, on average, of the growth-fragmentation. We present some Lyapunov-type conditions for this to hold. This is joint work with Denis Villemonais (Ecole des Mines de Nancy/Universite de Lorraine).

Spectral methods and their applications in the theory of branching random walks

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Keywords: branching random walks, multidimensional lattices, Green's functions, martingales, limit theorems.

In the modern theory of stochastic processes and their applications, the martingale methods are commonly understood as a wide range of probabilistic and analytical techniques based on the concept of a martingale (i.e. a process whose prediction in the "future", which is based on the "past", depends only on the "current state"). In the talk we examine an application of the theory of martingale and spectral analytical methods to a number of problems in such an actively developing field of stochastic processes as the theory of branching random walks (BRWs). Using of BRWs makes it possible to study the evolution of particle systems that can not only give offspring or die, but also walk on multidimensional lattices under various assumptions on environments according to rules that take into account "randomness" of a process see, e.g., [1]–[5]. We propose new methods for the study of BRWs based on a combination of martingale technique (see, Smorodina and Yarovaya, 2022) and the spectral theory which allowed, firstly, to expand the class of studied BRWs [6], secondly, to prove new limit theorems on the convergence in the mean square of some functionals, which defined on the trajectories of the studied processes [7], and, thirdly, to study the how the asymptotic behavior of BRWs depend on the structure of the spectrum of operators that determine the processes of walks and branching.

The study has been carried out at Steklov Mathematical Institute of Russian Academy of Sciences, and was supported by the Russian Science Foundation (RSF), project no. 23-11-00375.

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Branching Processes with Migration Subordinated by Renewal Process

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Keywords: Controlled branching process; renewal process; regenerative process; random time change; branching process with migration.

In [1], Nikolay Yanev introduced control branching processes with random control functions, known as φ-branching processes. Available results for this general class of processes, were presented in [2]. Recently, φ-branching processes with continuous time were introduced as well as limit theorems obtained in [3].

Using a renewal process as subordinator, we study branching processes with migration in continuous time. For these processes, we derive limit theorems assuming the offspring mean is one (critical case) and the emigration prevails over the immigration on average.

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Theta positive branching processes in varying environment

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Keywords: branching process, varying environment, theta branching, inhomogeneous Markov process.

This talk is based on the joint work [1] dealing with a time inhomogeneous Markov branching process $\{Z_t\}_{t\geq 0}$ with $Z_0 = 1$. It is a stochastic model for the fluctuating size of a population consisting of individuals that live and reproduce independently of each other, provided that the coexisting individuals are jointly effected by the shared varying environment in the following way:

- an individual alive at time t dies during the time interval $(t, t + \delta)$ with probability $\lambda_t\delta + o(\delta)$ as $\delta \to 0$,
- an individual dying at the time t is instantaneously replaced by k offspring with probability $p_t(k)$, where $k = 0$ or $k \geq 2$.

The time-dependent reproduction law of this model is summarized by two functions

$$\Lambda_t = \int_0^t \lambda_u du, \quad h_t(s) = p_t(0) + p_t(2)s^2 + p_t(3)s^3 + \dots,$$

where $h_t(s)$ is the probability generating function for the offspring number and Λ_t, assumed to be finite for all $t \geq 0$, is the cumulative hazard function of the life length of the initial individual.

In terms of the mean offspring number $a_t = \frac{\partial h_t(s)}{\partial s}|_{s=1}$, also assumed to be finite for all $t \geq 0$, the mean population size $\mu_t = \mathrm{E}(Z_t)$ has the following expression

$$\mu_t = exp\left\{\int_0^t (a_u - 1)d\Lambda_u\right\}.$$

Putting $m_t = \mathrm{E}(Z_t|Z_t{>}0)$, observe that $\mu_t = m_t\mathrm{P}(Z_t > 0)$. Let $q = \lim \mathrm{P}(Z_t{=}0)$ as $t \to \infty$ be the extinction probability of the branching process.

Compared to the time homogeneous setting, the added feature of varying environment makes the model very flexible and therefore cumbersome to study in the most general setting. In this paper, we distinguish between five classes of the branching processes in variable environment

(i) *supercritical* if $q < 1$ and $\lim \mu_t = \infty$,
(ii) *asymptotically degenerate* if $q < 1$ and $\liminf \mu_t < \infty$,
(iii) *critical* if $q = 1$ and $\lim m_t = \infty$,
(iv) *strictly subcritical* if $q = 1$ and $\lim m_t \in [1, \infty)$,
(v) *loosely subcritical* if $q = 1$ and $\lim m_t$ does not exist.

The division into five classes (i)-(v) is a modified version of the classification suggested in [2] for the branching processes in variable environment with discrete time. In [2], the classes (iv) and (v) are considered as one class called subcritical.

Paper [1] focuses on a special family of branching processes in variable environment which we call *theta-positive branching process*, cf [3], with the branching parameter $\theta \in (0, 1]$ in varying environment $(\{\lambda_t\}, \{a_t\})$. The branching parameter θ controls the higher moments of the offspring distribution specified by the formula

$$h_t(s) = 1 - a_t(1 - s) + a_t(1 + \theta)^{-1}(1 - s)^{1+\theta}.$$

It is assumed that the fluctuations of the mean offspring number at are restricted to a fixed interval

$$0 \leq a_t \leq 1 + 1/\theta.$$

The key feature of the theta-positive branching process Z_t is the explicit probability generating function

$$\mathrm{E}(s^{Z_t}) = 1 - (\mu_t^{-\theta}(1 - s)^{-\theta} + B_t(\theta))^{-1/\theta},$$

where $B_t(\theta) = \theta(1 + \theta)^{-1} \int_0^t \mu^{-\theta} a_u d\Lambda_u$.

For the presentation purposes, we consider the important special case of (2) with $\theta = 1$, when $p_t(0) = 1 - a_t/2$ and $p_t(2) = a_t/2$, the theta-positive branching process turns into the classical birth and death process in varying environment. In this case the generating function is linear-fractional

$$\mathrm{E}(s^{Z_t}) = 1 - \frac{\mu_t(1 - s)}{1 + \mu_t(1 - s)B_t},$$

with $B_t = \frac{1}{2} \int_0^t \mu_u^{-1} a_u d\Lambda_u$.

The following theorems presents our main results in the case $\theta = 1$ in terms of $V_t = \frac{1}{2} \int_0^t \mu_u^{-1} d\Lambda_u, \quad V = \lim V_t, \quad \Lambda = \lim \Lambda_t$.

Theorem 1. If $V < \infty$, then q < 1, $\lim \mu_t = \mu$, $0 < \mu \leq \infty$, and $q = (V + \frac{1}{2} + \frac{1}{2}\mu^{-1})^{-1}$. If $V = \infty$, then q $= 1$ and $\mathrm{P}(Z_t > 0) \sim (V_t + \frac{1}{2}\mu^{-1})^{-1}, \quad m_t \sim \mu_t V_t + \frac{1}{2}$.

Theorem 2. A theta-positive branching process is supercritical if and only if $V < \infty$, and $\Lambda = \infty$. In this case, $\lim \mu_t = \infty$, $q = (V + \frac{1}{2})^{-1}$ and $\mu_t^{-1} Z_t$ almost surely converges to a random variable W such that $\mathrm{E}(e^{-wW}) = 1 - (V + \frac{1}{2} + 1/w)^{-1}$.

Theorem 3. A theta-positive branching process is asymptotically degenerate if and only if $\Lambda < \infty$. In this case, $\lim \mu_t = \mu, \quad 0 < \mu < \infty$ and Z_t almost surely converges to a random variable Z_∞ such that $\mathrm{E}(s^{Z_\infty}) = 1 - 1/(V + \frac{1}{2}(1 - 1/\mu) + 1/(\mu(1 - s)))$.

Corollary 1. If $\Lambda < \infty$ and $a_t \equiv 0$, then the theta-positive branching is asymptotically degenerate with $\mu = e^{-\Lambda}$ and $\mathrm{E}(s^{Z_\infty}) = 1 - \mu + \mu s$.

Theorem 4. A theta-positive branching process is critical if and only if $V = \infty$ and $\mu_t V_t \to \infty$. In this case, $\mathrm{P}(Z_t > 0) \sim 1/V_t, \quad m_t \sim \mu_t V_t$ and $\lim \mathrm{E}(e^{-wZ_t/m_t}) = 1 - 1/(1 + 1/w), \quad w \le 0$.

Corollary 2. If $\Lambda = \infty$ and $0 < \liminf \mu_t \le \limsup \mu_t < \infty$, then the theta-positive branching is critical.

Theorem 5. A theta-positive branching process is strictly subcritical if and only if $V = \infty$ and $\mu_t V_t \to M, \quad 0 \le M \le \infty$. In this case, $\mu_t \to 0$, $\mathrm{P}(Z_t > 0) \sim m\mu_t, \quad m_t \sim m, \quad m = M + \frac{1}{2}$ and $\mathrm{E}(s^{Z_t}|Z_t > 0) \to 1 - m/(M - \frac{1}{2} + 1/(1 - s))$.

Theorem 6. A theta-positive branching process is loosely subcritical if and only if $V = \infty$ and $\mu_t V_t$ does not have a limit. In this case, there are several subsequences $t' = \{t_n\}$ leading to different partial limits $\mu_t V_t \to M, \quad t' \to \infty$.

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Branching selfdecomposability and limit theorems for superposition of point processes

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Keywords: point process, continuous branching, limit theorems, superposition, selfdecomposability, stability.

Limit theorems for superposition of independent point processes (PPs) must involve an operation that makes them ”thinner” so that a limit of superposition of their growing number exist. It is analogous to scaling for random variables, but to preserve a PP framework, this "scaling"of a PP must be stochastic acting independently on the PP' points. We show that the most general such operation on a PP is independent branching of its points with diffusion. The simplest example is given by pure-death process without displacement of points which is equivalent to independent thinning of points. Given such an operation, one can formulate limit theorems for superposition of independent PPs aiming to characterise all possible limits. The processes which may arise as a limit are selfdecomposable (SD) PPs which are a strict subclass of infinitely divisible (ID) PPs. At the same time, it is strictly larger than the class of strictly stable PPs which arise as a limit of scaled superposition of i.i.d. PPs. Since SD PPs are also ID, their distribution is characterised by Levy measure (also known as KLM measure in PP context) and it has a special integral representation from potential theory and the theory of general Markov processes. We fully characterise the Levy measures of SD PPs and prove their series decomposition which is a generalisation of LePage series known for stable PPs [1, 2] and which mimics the stochastic integral representation of SD random variables.

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Path 2

Stochastic analysis

Estimation of the Parameter of one Class of Distributions

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Keywords: Random variables, Order Statistics, Estimating the Parameter of Distribution.

Let $X_1, X_2, ..., X_n$ be independent, non-negative and identically distributed random variables (r.v.) with common distribution function (d.f.) $F(x)$. $X_n^{(1)}, X_n^{(2)}, ..., X_n^{(n)}$ are order statistics of r.v $X_1, X_2, ..., X_n$, arranged in descending order, i.e. $X_n^{(1)} \geq X_n^{(2)} \geq ... \geq X_n^{(n)}$. We introduce the following class of d.f. $F(x)$ defined on $(0, \infty)$:

$$R_\alpha = \left\{ F(x) : 1 - F(x) = \exp\left(-x^{\frac{1}{\alpha}} L(x)\right), x \geq x_0 > 0 \right\},$$

where $0 < \alpha < \infty$ and

$$L(x) = \exp\left\{ \int_\alpha^x \frac{\varepsilon(t)}{t} dt \right\}, a > 0, \varepsilon(t) \to 0 \text{ as } t \to \infty.$$

In what follows, we will assume that parameter α entering the definition of this class is unknown. For estimating parameter α of this class, the author in [1] proposed and studied estimates involving k extremal order statistics or all elements of order statistics with weight coefficients. This article is a continuation of studies conducted in [1]; an estimate, which consists of one element of the order statistics for the following parameter is proposed:

$$\alpha_{k,n} = \frac{\log X_n^{(k)}}{\log \log \frac{n}{k}}, \text{ for } 1 < k < n$$

We assume that the d.f. $F(x)$ is strictly increasing and continuously differentiable. The main result of this article is reduced to the following theorem.

Theorem. Let $k \to \infty$ as $n \to \infty$ such that $k = o(n)$ and

$$\sqrt{k} \left(\left(\log \frac{n}{k} \right)^\alpha \right) \log \frac{n}{k} \log \log \frac{n}{k} \to 0$$

Then as $n \to \infty$

$$\sqrt{k} \log \frac{n}{k} \log \log \frac{n}{k} \alpha^{-1} \left(\alpha_{k,n} - \alpha + b_{k,n}(\alpha) \right) \xrightarrow{dis} N(0,1),$$

where symbol $\xrightarrow{dis}$ means convergence by distribution, $N(0,1)$ are standard Normally distributed r.v. and

$$b_{k,n}(\alpha) = \frac{\log\left(\log^\alpha \frac{n}{k}\right)}{\log \log k} \to 0 \quad \text{as} \quad n \to \infty.$$

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Digits of Powers of 2 in Ternary Numeral System

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Keywords: Ternary number system, Benford's law, powers of two.

We study the digits of the powers of 2 in the ternary number system. We propose an algorithm for doubling numbers in ternary number system. Using this algorithm, we explain the appearance of blocks of 2s and 0s when the number 2^{n+1} is written on top of 2^n $(n = 0, 1, 2, ...)$ in a natural way so that for example the last digits are forming one column, the second to the last digits are forming another column, and so forth. We also look at the patterns formed by the first digits, the patterns formed by the last digits and use this to prove that the sizes of these blocks of 0s and 2s are unbounded. We also discuss how this regularity changes when the digits move from left end of the numbers to the right end. Let us write first powers of two $(1, 2, 4, ..., 2^{15})$ in ternary numeral system so that their digits in the corresponding place values are aligned along vertical columns.

1	1	**2**	**2**	**2**	**2**	1	1	**2**	**2**
	2	1	1	1	1	0	**2**	1	1
	1	0	**2**	0	**2**	0	1	0	**2**
		1	**2**	1	**2**	1	**2**	0	1
			2	**2**	1	0	**2**	1	**2**
			1	1	0	1	**2**	**2**	1
				2	0	0	**2**	**2**	**2**
				1	0	0	1	1	1
					1	1	**2**	0	**2**
						2	1	0	1
						1	0	1	**2**
							1	**2**	1
								2	**2**
								1	1
									2
									1

We discuss the patterns occurring in base 3 representation of powers of 2. We show that

1. every string of ending digits appears infinitely often, provided the string does not end in 0,
2. every string of starting digits (not beginning with 0) appears infinitely often,
3. if the powers of 2 are all written in base 3 as one column so that the digits of the same place value are on top of each other, then the size of the triangular blocks of zeros and twos grow indefinitely.

Part 1 was proved using number theory methods, but it can also be proved using the fact that 2 is a primitive root modulo each power of 3. Part 2 was proved using the fact that for irrational number α, the sequence $x_n = \{n\alpha\}$, where $\{x\}$ is the fractional part of x, is uniformly distributed in $[0, 1]$, but it can also be interpreted in the context of "Benford's law". Part 3, which is the main objective of the current study, is shown to be a direct consequence of the Parts 1 and 2. Also, the change of the distribution of probabilities of these combination of digits when they are taken in between the left and right endpoints, is studied.

Theorem 1. If the powers of 2 are written so that each next power of 2, in ternary number system notation, is written on top of the previous power of 2, and the digits corresponding to the same place values are all on the same vertical lines, then arbitrarily large triangular blocks of zeros (twos) can appear in this infinite triangular table.

The current study is related to the specific question asked by P. ErdE‘s: how frequently do the powers of 2 have ternary expansions that omit the digit 2? He conjectured that this holds only for finitely many powers of 2. See [1] for the discussion of this problem. In view of the results of the current study, ErdE‘sвЂ™s conjecture can be interpreted in the way that there are only finitely many powers of 2 which does not intersect the blocks containing only twos.

Theorem 2. The probability of an m-digit number A, which may or may not start with zero digit, appearing at kth position ($k > 1$) from left of 3-base representations of 2^n, is

$$p_k(A) = \log_3 \prod_{i=3^k}^{3^{k+1}-1} \left(1 + \frac{1}{3^m i + A}\right).$$

The obtained result generalize the results given in [2,3] about the significant digit phenomenon also known as The Significant-Digit Law. In particular, this means that a sequence of m zeros 00...0 is more likely to appear towards the left side of the table than a sequence of m twos 22...2, but as the block of m digits approach the right side of the construction then the probabilities become more unified.

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Central limit theorem for stochastic perturbations of PL circle maps with two break points

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Keywords: Central limit theorem, circle dynamics, stochastic perturbation, markov chain.

One of the classical problems of the theory dynamical systems is asymptotic behavior of stochastic perturbations of one-dimensional dynamics. J. Crutchfield et al. [1] and B. Shraiman et al. [2] heuristically considered in their works a renormalization group-respectively a field theoretic path-integral approach for weak Gaussian noise perturbing one dimensional maps with period doubling at the onset of chaos. E.Vul, Y.Sinai, and K.Khanin [3] studied the effect of noise on the ergodic properties of these maps and showed that for systems with weak noise at the accumulation of period doubling, there is a stationary measure, depending on the magnitude of the noise, which converges for vanishing noise to the invariant measure of the attractor. O.Espinosa and R.Llave [4] studied stochastic perturbations of several systems using the renormalization group technique and proved a central limit theorem for critical circle maps with a golden mean rotation number and some mild conditions on the stochastic noise. A. Dzhalilov, D. Mayer and A. Aliyev [5] investigated this problem for circle maps with a break point. We extend these results for piecewise linear (PL) circle maps with bounded type irrational rotation number.

Now, we turn to the formulation of the main results of our work.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be a PL circle homeomorphism. Let the stochastic sequence defined as

$$\bar{x}_{n+1} = T(\bar{x}_n) + \sigma\xi_{n+1},\ \bar{x}_0 := x \in S^1$$

where (ξ_n) be a sequence of independent random variables with $p > 2$ finite moments satisfying following conditions:

$$E\xi_n = 0 \tag{1}$$

$$const \leq (E|\xi_n|^2)^{\frac{1}{2}} \leq (E|\xi|^p)^{\frac{1}{p}} \leq Const. \tag{2}$$

Let $\omega_n(x,\sigma)$ be the stochastic process defined by

$$\omega(x,\sigma_n) = \frac{\bar{x}_n - x_n}{\sigma_n\sqrt{\operatorname{var}(\bar{x}_n)}} \tag{3}$$

We formulate the main result of this work.

Theorem. *Let T be a piecewise linear circle homeomorphism with bounded type irrational rotation number and two break ponts b_1 and b_2 on different orbits. Consider a sequence of independent random variables (ξ_n) with $p > 2$ finite moments satisfying the conditions (1) and (2). Then there exist a constant $\gamma > 1$ depending on T and p, for all sequences σ_n satisfying the condition:*

$$\lim_{n\to\infty} \sigma_n n^\gamma = 0,$$

the process $\omega_{q_n}(x,\sigma_{q_n})$ converges in distribution to the standard Normal random variable.

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SIS model on (random) dense graphs: probabilistic approach and remarks on optimal vaccinations

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Keywords: heterogeneous SIS model, individual based model, reproduction number, vaccination, herd immunity, equilibrium.

We consider an individual based model on a random dense graph for a SIS epidemic in an heterogeneous population of size n :

- Each individual i has a feature (age, localization, ...), say $x_i \in X$.
- Individuals i and j are connected with probability $w_E^{(n)}(x_i, x_j) \in [0,1]$.
- Individual i is either Susceptible (S) or Infectious (I), and denote by E_t^i its state at time $t \geq 0$.
- Individual j infects individual i at rate $w_I^{(n)}(x_i, x_j) \geq 0$ (provided that i is S, j is I and i, j are connected).
- Individual i recover at rate $\gamma(x_i) > 0$ (provided that i is I).

The infected population at time t is described by the random measure:

$$\rho_t^{(n)}(dx) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{E_i^t = I} \delta_{x_i}(dx).$$

Under suitable conditions (see [1]), the random measure $\rho_t^{(n)}$ converges to a deterministic limit $\rho_t(dx) = u(t,x)\mu(dx)$, with $u(t,x)$, the infected fraction at time t of individuals with feature x, unique solution to:

$$\boxed{\partial_t u(t,x) = (1 - u(t,x)) \int_{x' \in X} k(x,x') u(t,x') \mu(dx') - \gamma(x) u(t,x),}$$

where $\mu(\mathrm{dx}')$ is the (asymptotic) distribution of the feature x' in in the population and:

$$k(x, x') = \lim_{n\to\infty} n w_E^{(n)}(x, x') w_I^{(n)}(x, x').$$

The proof use a coupling argument with a SIS model on a complete graph.

The reproduction number R_0 is then the spectral radius of the integral operator with kernel $k(x, x')/\gamma(x')$, see [2]. If $R_0 \geq 1$, 0 is the only equilibrium of the ODE and $\lim_{t\to\infty} u(t, \cdot) = 0$; if $R_0 > 1$ then there exists a non-zero maximal equilibrium, say g^*.

The use of a perfect vaccine at time $t = 0$, can be interpreted as replacing the population distribution $\mu(dx)$ by the effective population distribution $\eta(x)\ \mu(dx)$, where $\eta(x)$ is the fraction of un-vaccinated population with feature x, see [2,3]. The corresponding effective reproduction number $R_e(\eta)$ is the spectral radius of the integral operator with kernel $k(x, x')\eta(x')/\gamma(x')$. In particular, vaccinating uniformly the population with probability $1 - 1/R_0$ (that is $\eta^{\mathrm{unif}} = 1/R_0$) is critical as $R_e(\eta^{\mathrm{unif}}) = 1$ (so that the epidemic vanishes asymptotically). We give a rigorous proof in [4] of the intuitive fact that vaccinating a fraction $g^*(x)$ of the population with feature x, that is $\eta = 1 - g^*$, is critical:

$$\boxed{R_e(1 - g^*) = 1.}$$

This is joint work with D. Dronnier, P. Frasca, F. Garin, V. C. Tran, A. Velleret and P.-A. Zitt.

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Note on critical mappings of the circle with rotation number of algebraic type

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Keywords: circle homeomorphisms, rotation number, hiting time.

Consider the space $X_{cr}(\rho)$ of real-analytic homeomorphisms of the circle with rotation number $\rho = [k_1, k_2, k_1, k_2, ...]$, $k_1, k_2 \in N$. And with one critical point at which the derivative is early zero. It is well known (see [1]) that the renormalization transformation $R_{cr} = R_{cr}(k_1, k_2)$ into $X_{cr}(\rho)$ has a single fixed point $T_{cr} = T_{cr}(k_1, k_2)$. Denote by $Cr(T_{cr})$ the set of all critical C^1-conjugate to T_{cr} maps of the circle.In the work (see [2], [3]) a unique pair (U_{k_1}, U_{k_2}) of potentials was constructed corresponding to all mappings $X_{cr}(\rho)$ from.The main goal of this work is the behavior of the normalized hit times in small neighborhoods of the critical point.

The distribution function $\Phi_n^{(1)}(t)$ and the corresponding first hit function $E_n^{(1)}(t)$ can be expressed using the lengths of segments of dynamic partition elements P_n.

We now formulate the main result of our work.

Theorem. Consider the critical mapping $T \in Cr(T_{cr})$. And the sequence of distribution functions $\left\{\Phi_n^{(1)}(t)\right\}_{n=1}^{\infty}$ corresponding to the normalized function of the first hit $\overline{E}_n^{(1)}(x)$ in the $n-$th renormalization neighborhood of the singular point x_0.

Then

1) For all $t \in R^1$ there are finite limits and the following relations are true:

$$\lim_{n\to\infty} \Phi_{2n-1}^{(1)}(t) = \Phi_{k_1}^{(2)}(t),\ \lim_{n\to\infty} \Phi_{2n}^{(1)}(t) = \Phi_{k_2}^{(2)}(t);$$

2) $\Phi_{k_1}^{(2)}(t) = 0,\ \Phi_{k_2}^{(2)}(t) = 0$, if $t \leq 0$ and $\Phi_{k_1}^{(2)}(t) = 1,\ \Phi_{k_2}^{(2)}(t) = 1$, if $t \geq 1$

3) Both functions $\Phi_{k_1}^{(2)}(t)$ and $\Phi_{k_2}^{(2)}(t)$ are continuous on the line.

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Convergence rate estimates in the Hartman-Wintner Law of the Iterated Logarithm

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Keywords: law of the iterated logarithm, convergence of a series, number of exits, normal law, rate of convergence in the central limit theorem

The work is devoted to further refinement of the classical Hartman-Wintner theorem on the law of the iterated logarithm for a sequence of independent, identically distributed random variables. Namely, an estimate of the rate of convergence in the form of convergent series of weighted probabilities of large deviations is established – the exact asymptotes in the small parameter of the series, which is a refinement of the corresponding result [1]. Analogs of the obtained results were proved for a family of independent, identically distributed random variables indexed on sectors of the d-dimensional lattice of the Euclidean space

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Quadratic Stochastic Operators with Countable State Space

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Keywords: regular transformation, Volterra quadratic stochastic operator, infinite state space.

Let $(X, \mathcal{F})$ be a measurable space, where X is a state space and $\mathcal{F}$ is σ-algebra on X, and $S(X, \mathcal{F})$ be the set of all probability measures on $(X, \mathcal{F})$. Let $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ be a family of functions on $X \times X \times \mathcal{F}$ such that $P(x, y, \cdot) \in S(X, \mathcal{F})$, where $P(x, y, A)$ be regarded as a function of two variables x and y with fixed $A \in \mathcal{F}$, is a measurable function on $(X \times X, \mathcal{F} \otimes \mathcal{F})$ and $P(x, y, A) = P(y, x, A)$ for any $x, y \in X$ and $A \in \mathcal{F}$.
Specifying such family of functions $\{P(x, y, A) : x, y \in X, A \in \mathcal{F}\}$ we introduce a non-linear transformation $V : S(X, \mathcal{F}) \to S(X, \mathcal{F})$ which is defined by

$$(V\lambda)(A) = \int_X \int_X P(x, y, A) d\lambda(x) d\lambda(y),$$

where $A \in \mathcal{F}$ is an arbitrary measurable set.
The case with finite state space X was considered by Bernstein [1].
When state space X is an infinite countable set of positive integers, and $\mathcal{F}$ is the power set $\mathcal{P}(N)$ of N, i.e. the set of all subsets of N, then

$$S^N = \{\mathbf{x} = (x_i)_{i=1}^{\infty} : \forall i \;\; x_i \geq 0, \sum_{i=1}^{\infty} x_i = 1\}$$

is the set of all probability measures on $(N, \mathcal{P}(N))$.
In this case, the measure $P(i, j, \cdot)$ is a discrete probability measure and the corresponding qso $V : S^N \to S^N$ has the following form

$$(V\mathbf{x})_k = \sum_{i,j=1}^{\infty} P_{ij,k} x_i x_j, \quad k \in N \tag{1}$$

where the coefficients $P_{ij,k}$ satisfy the following conditions:

$$a) P_{ij,k} \geq 0; \quad b) P_{ij,k} = P_{ji,k}; \quad c) \sum_{k=1}^{\infty} P_{ij,k} = 1 \text{ for all } i, j, k \in N.$$

The quadratic stochastic operator V (1) is called Volterra, if $p_{ij,k} = 0$ for any $k \notin \{i, j\}$.
The biological treatment of such operators is rather clear: the offspring repeats one of its parents.
A qso V is a Volterra if and only if

$$(V\mathbf{x})_k = x_k(1 + \sum_{i=1}^{\infty} a_{ki} x_i)$$

where $A = (a_{ij})_1^\infty$ is a skew-symmetric matrix with $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$, $a_{ii} = 0$ and $|a_{ij}| \leq 1$. Here $i, j \in N$.
A qso V is called a regular if for any initial point $\mathbf{x} \in S^N$, a limit

$$\lim_{n\to\infty} V^n(\mathbf{x})$$

exists.
Note that the limit point is a fixed point of a qso V. Thus the fixed points of qso describe a limit or a long run behavior the trajectories at any initial point. A limit behavior of trajectories and the fixed points of qso play an important role in many applied problems [2],[3].
A nonlinear operator V defined on the finite-dimensional simplex S^{m-1} is called ergodic if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x)$$

exists for any $\mathbf{x} \in S^{m-1}$. It is evident that a regular qso V is ergodic; however, regularity does not follow from the ergodicity
On the basis of numerical calculations, Ulam conjectured [4] that an ergodic theorem holds for any qso V. In 1977, Zakharevich [5] proved that in general this conjecture is false. He considered the following Volterra operator on S^2

$$\begin{aligned} x_1' &= x_1(1 + x_2 - x_3) \\ x_2' &= x_2(1 - x_1 + x_3) \\ x_3' &= x_3(1 + x_1 - x_2) \end{aligned}$$

and proved that it is a non-ergodic transformation.
In this paper we consider the limit behaviour of the trajectories of the following Volterra operator with infinite state space as follows.

$$\begin{aligned} &(V\mathbf{x})_1 = x_1[1 + ax_2 - ax_3 + ax_4 - ax_5 + ax_6 - ax_7 + \cdots] \\ &(V\mathbf{x})_2 = x_2[1 - ax_1 + ax_3 - ax_4 + ax_5 - ax_6 + ax_7 - \cdots] \\ &\cdots \\ &\cdots \\ &(V\mathbf{x})_{2n-1} = x_{2n-1}[1 + ax_1 - ax_2 + \cdots - ax_{2n-2} + ax_{2n} - ax_{2n+1} + ax_{2n+2} - ax_{2n+3} + \cdots] \\ &(V\mathbf{x})_{2n} = x_{2n}[1 - ax_1 + ax_2 - \cdots - ax_{2n-1} + ax_{2n+1} - ax_{2n+2} + \cdots] \\ &\cdots \\ &\cdots \end{aligned}$$

where $a \in [-1, 1]$.

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Long-term behaviour of the neutron transport equation at criticality

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Keywords: neutron transport, criticality, asymptotic moments, Perron-Frobenius, Yaglom, survival probability

The neutron transport equation (NTE) describes the flux of neutrons over time through an inhomogeneous fissile medium. Understanding the long-term behaviour of solutions to the NTE is vital for nuclear safety and reactor design. One way to do this is to consider the so called k-effective eigenvalue problem. The eigenvalue, k_{eff}, has the physical interpretation as being the ratio of neutrons produced (during fission events) to the number lost (due to absorption in the reactor or leakage at the boundary) per typical fission event and determines the criticality of the system. In this talk, we will prove the existence of k_{eff} and the corresponding eigenfunctions via a Perron Frobenius type result by modelling the nuclear fission process as an appropriate discrete time branching process. We will then focus on the critical case ($k_{\text{eff}} = 1$) and discuss the limiting behaviour of the process.

This talk is based on joint work with Alex M. G. Cox, Eric Dumonteil, Andreas Kyprianou, Denis Villemonais and Andrea Zoia.

Weakly Dependent Properties of Vertex Processes of a Convex Hull

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Keywords: Convex Hull, Binomial Point Process, Poisson Point Process, Vertex Processes.

The work is devoted to the study of the properties of convex hulls generated by the implementation of a inhomogeneous Poisson point process on the unit disk.

Efron B. [1], Reny A. and Sulanke R. [2] and other researchers were the first to study the functionals of the convex hull; they found the asymptotic behavior and revealed the connections between the asymptotic expressions for the mathematical expectations of the number of vertices, area and perimeter of the convex hull in the case when random points are uniformly distributed in the square. Carnal H. in [3], obtained asymptotic expressions for similar convex hull functionals generated by random points set in polar coordinates. Their components are independent of each other, the angular coordinate is uniformly distributed, and the tail of the distribution of the radial coordinate is a regularly varying function near the boundary of the support - the disk or at infinity.

Using the approximation of a binomial point process to a homogeneous Poisson process, Groeneboom P. in [4], managed to prove the central limit theorem for the number of vertices of a convex hull, for the case when the support of the original uniform distribution is a convex polygon or ellipse.

In this article, the property of strong mixing and the martingale property of functionals of vertex processes of the convex hull are established, in the case when the convex hull is generated from a inhomogeneous Poisson point process inside the disk.

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On the distribution of the crossing number of a strip by trajectories of the Levy process

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Keywords: stochastic Levy process, crossing number of a strip, asymptotic formulas, probabilistic inequalities

Let $\xi(t)$, $t \geq 0$, $\xi(0) = 0$, be a Levy process, i.e., a homogeneous stochastic process with independent increments whose sample functions are continuous on the right. For it we have

$$\mathbf{E}\exp\{\lambda\xi(t)\} = \exp\{t\psi(\lambda)\}, \qquad \psi(\lambda) = \gamma\lambda + \frac{\sigma^2\lambda^2}{2} + \int\limits_{-\infty}^{\infty}\left(e^{\lambda x} - 1 - \frac{\lambda x}{1+x^2}\right)dS(x), \quad (1)$$

with standard conditions on γ, σ and $S(x)$. Introduce two sequences of stopping times: $\tau_0^+ = \tau_0^- = 0$,

$$\tau_i^- = \inf\{t > \tau_{i-1}^+ : \ \xi(t) \le -a\}, \qquad \tau_i^+ = \inf\{t > \tau_i^- : \ \xi(t) \ge b\}, \qquad i \ge 1,$$

where $a > 0$, $b > 0$. Define a random variable θ equal to the number of upcrossings of the strip $\{-a < y < b\}$ on the plane (x, y) by trajectories of the process $(t, \xi(t))_{t=0}^{\infty}$. This variable is finite with probability one if $\mathbf{E}\xi(1) \ne 0$.

Our goal is to study the distribution of θ. The exact calculation of it is available only in some particular situations. The talk is devoted to finding asymptotic formulas for $\mathbf{P}(\theta \ge t)$ as $b \to \infty$ and estimates for this probability.

It is easily seen that

$$\mathbf{P}(\theta \ge k) = \mathbf{P}(\tau_k^+ < \infty), \qquad k \ge 1.$$

I. Asymptotic formulas.

Suppose that $\mathbf{E}\xi(1) < 0$ and put, for $\operatorname{Re} l = 0$, $x \ge 0$, and $y \le 0$,

$$\zeta = \sup_{t \ge 0} \xi(t), \qquad Q(x) = \mathbf{P}(\zeta \ge x),$$

$$\eta_-(y) = \inf\{t \ge 0 : \ \xi(t) \le y\}, \qquad \chi_-(y) = \xi(\eta_-(y)) - y, \qquad \mathbf{P}(\chi_- < t) = \lim_{y \to -\infty} \mathbf{P}(\chi_-(y) < t).$$

Asymptotic behavior of $\mathbf{P}(\theta \ge k)$ is based on the asymptotic properties of ζ and χ_-. Let

$$\rho = \sup\{\lambda \ge 0 : \ \mathbf{E}\exp\{\lambda \xi(1)\} \le 1\} = \sup\{\lambda \ge 0 : \ \psi(\lambda) \le 0\}.$$

The following result is known ([1]).

Lemma 1. *Suppose that* $\rho > 0$, $\quad \psi(\rho) = 0$, $\quad \psi'(\rho) = \mathbf{E}\,\xi(1)\, e^{\rho\xi(1)} < \infty$. *Then*

$$Q(x) = c\, e^{-\rho x}(1 + o(x)), \qquad x \to \infty,$$

where

$$c^{-1} = \rho\psi'(\rho) \int_{-\infty}^{0} e^{\rho y}\, dF_-(y), \qquad F_-(y) = -\int_0^{\infty} \mathbf{P}(\inf_{0 \le s \le t} \xi(s) \ge y)\, dt, \qquad y \le 0.$$

Theorem 1. *Under conditions of lemma 1 we have*

$$\mathbf{P}(\theta \ge k) = \mathbf{P}(\theta \ge 1)(hc)^{k-1} e^{-\rho(k-1)(a+b)}(1 + o(1)), \qquad b \to \infty,$$

for all $k \ge 2$ *and arbitrary* $a > 0$, $h = \mathbf{E}\, e^{\rho\chi_-}$. *If, in addition,* $a \to \infty$ *then for all* $k \ge 1$

$$\mathbf{P}(\theta \ge k) = (hc)^k e^{-\rho k(a+b)}(1 + o(1)).$$

II. Inequalities.

The estimates of this section can be considered as a natural addition to the asymptotic results. Similar inequalities for random walks generated by sums of i.i.d. random variables can be found in [2].

The following inequalities hold for all $k \ge 1$.

Theorem 2. *Let* $\mathbf{E}\,\xi(1) < 0$. *Then*

(1) $\mathbf{P}(\theta \geq k) \leq Q^k(a+b)$.

(2) *If* $\int\limits_{-\infty}^{-r} d\,S(x) = 0$ *for some* $r \geq 0$ *then* $\mathbf{P}(\theta \geq k) \geq Q^k(a+b+r)$.

Theorem 3. *Suppose that* $\mathbf{E}\,\xi(1) < 0$, $\mathbf{E}\,|\xi^3(1)| < \infty$ *and the function* $Q(x)$ *is convex for* $x > 0$. *Then*

$$\mathbf{P}(\theta \geq k) \geq Q^k(a+b+l), \qquad \text{where } \ l = 3a_3/a_2, \qquad a_i = \int_{-\infty}^{\infty} |x^i|\,d\,S(x), \qquad i = 2, 3.$$

Theorem 4. *Suppose that* $\mathbf{E}\,\xi(1) < 0$, $\mathbf{E}\,|\xi^3(1)| < \infty$ *and* $\mathbf{E}\,\exp\{\rho\xi(1)\} = 1$ *for some* $\rho > 0$. *Then*

$$\mathbf{P}(\theta \geq k) \geq s^{-1}\,e^{-\rho k(a+b+l)},$$

where $s = \sup_{0<x<M} \mathbf{E}\big(e^{\rho(\xi(1)-x)}|\xi(1) \geq x\big)$, $M = \inf\{x > 0: \ \mathbf{P}(\xi(1) \leq x) = 1\}$.

Theorem 5. *Suppose that* $\mathbf{E}\,\xi(1) < 0$, $\mathbf{E}\,|\xi^3(1)| < \infty$ *and* $\mathbf{E}\,\exp\{\lambda\xi(1)\} = \infty$ *for all* $\lambda > 0$. *Consider a process* $\xi_1(t)$ *which is obtained by replacing jumps of the process* $\xi(t)$ *that exceed a number* $q > 0$, *by jumps of size* q. *Thus, the spectral function of the representation (1) for the process* $\xi_1(t)$ *equals*

$$S_1(x) = \begin{cases} S(x), & \text{if} \quad x \leq q, \\ S(q), & \text{if} \quad x > q, \end{cases}$$

and the process $\xi_1(t)$ *satisfies conditions of the theorem* 4. *Then*

$$\mathbf{P}(\theta \geq k) \geq s_1^{-1}\,e^{-\rho_1 k(a+b+l)},$$

where s_1 *and* ρ_1 *are defined as above by using* $\xi_1(t)$ *instead of* $\xi(t)$, *the number* l *is defined in the theorem* 3.

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Boundedness of the classical operators in weighted quasi-Banach spaces of entire functions[1]

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Keywords: weighted spaces of holomorphic functions, weighted composition operator, Volterra operator, Bergman spaces, Fock spaces.

We consider a problem of boundedness of classical operators acting on weighted quasi-Banach spaces of holomorphic functions $H_v(G)$. This space is defined as follows:

$$H_v(G) = \{f \in H(G), \|f\|_v = \sup_{z\in G} \frac{|f(z)|}{v(z)} < \infty\}.$$

Here G is a domain in the complex plane, $H(G)$ is the space of all holomorphic functions in G, and v is a weight on G.

Here and bellow $X \hookrightarrow H(G)$ is a quasi-Banach space endowed with the quasi-norm $\|\cdot\|$. X^* is a dual space of X consisting of all linear continuous functionals on X endowed with the norm $\|\cdot\|^*$. δ_z is δ-function for a fixed point $z \in G$.

Theorem 1. *Let v be an arbitrary weight on G. Linear operator $T : X \mapsto H_v(G)$ is bounded if and only if*

$$a)\ \delta_z(T) \in X^* \ \text{ for all } \ z \in G; \quad b)\ \sup_{z\in G} \frac{\|\delta_z(T)\|^*}{v(z)} < \infty.$$

This result allowes to establish some criteria of the boundedness of weighted composition and Volterra operators an abstract quasi-Banach space in terms of δ-functions norms. As a consequence we obtain criteria of the boundedness of the above mentioned operators on generalized Bergman, Hardy and Fock spaces. In particular cases it is possible to state these criteria in terms of weights defining spaces and functions giving the composition operator. In comparison with the previous results (see [1]) we essentially extend the class of weighted holomorphic spaces in the unit disc that admits a realization of Zorboska's method. In addition, we develop an extension of this approach to weighted spaces of entire functions. In this relation we introduce the class of almost harmonic weights and obtain some estimates of δ-functions norms in spaces dual to the generalized Fock spaces giving by almost harmonic weights. Finally, these results are applied to classical growth spaces and the Cesáro operator acting on them.

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Stochastic longitudinal oscillations viscoelastic rope with moving boundaries, taking into account damping forces

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Keywords: linear mathematical model, vibrations of a rope, moving boundaries, stochastic vibrations

At present, reliability issues in the design of machines and mechanisms require more and more complete consideration of the dynamic phenomena that take place in the designed objects. The widespread use in technology of mechanical objects with moving boundaries necessitates the development of methods for their calculation. The problem of oscillations of systems with moving boundaries is related to obtaining solutions to integro-differential and partial differential equations in time-variable domains [1-10]. Such tasks are currently not well understood. Their peculiarity is the difficulty in using the known methods of mathematical physics, suitable for problems with fixed boundaries. The complexity of the solutions obtained is explained by the fact that up to now there has not been a sufficiently general approach to the analysis of the features of the dynamics of such systems. In connection with the danger of resonance, the study of forced oscillations is of great importance here. Attempts to investigate this process have been made, but the results obtained are limited mainly by a qualitative description of dynamic phenomena [1-4]. In addition, it is recognized that deterministic modeling of systems cannot be adequate for some types of problems, so it is necessary to switch to probabilistic-statistical, where there are random variables, stochastic fluctuations. When solving here, mainly approximate methods are used [5-9], since obtaining exact solutions is possible only in the simplest cases [10].

If the damping of transverse vibrations is mainly due to the action of external damping forces, then in the case of longitudinal vibrations, the damping is mainly affected by elastic imperfections in the material of the vibrating object [5-10]. The study of viscoelasticity includes the analysis of the stochastic stability of stochastic viscoelastic systems, their reliability, etc. The paper considers stochastic linear longitudinal oscillations of a viscoelastic beam with moving boundaries, taking into account the influence of damping forces. The case of a difference kernel makes it possible to reduce the problem of analyzing a system of stochastic integro-differential equations to the study of a system of stochastic differential equations. To estimate the expansion coefficients, it is proposed to apply the statistical numerical Monte Carlo method [11].

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Limited Theorem for Stochastic Integrals over Semi-Martingales

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Keywords: Martingale, Semi-martingale, Wiener process, Kolmogorov distance.

Let a sequence of semi-martingales $X^n = (X_s^n, \Im_s^n), X_0^n = 0, n \geq 1$ be given on some probability space $(\Omega, \Im, P)$ with selected filtration flows $F^n = (\Im_s^n)$ and satisfying the ordinary conditions.

Let, further, $W = (W_s, \Im_s)$ be a standard Wiener process (with respect to some flow $F = (\Im_s), s \geq 0$).

The main purpose of this article is to give an estimate for the Kolmogorov distance $R^n = \sup_{x \in R} |F^n(x) - F(x)|$ between distribution functions $F^n(x) = P\left(\int_0^1 f(s, X_s^n) dX_s^n \leq x\right)$ and $F(x) = P\left(\int_0^1 f(s, W(s)) dW(s) \leq x\right)$.

The estimate obtained in this article generalizes the results previously obtained by the author in [1] and [2] for the Kolmogorov distance $R_n = \sup_{x \in R} |G^n(x) - G(x)|$ between distribution functions $G^n(x) = P\left(\int_0^1 f(s, M_{s_-}^n) dM_{s_-}^n \leq x\right)$ and $G(x) =$

$P\left(\int_0^1 f(s,W(s))dW(s) \leq x\right)$, where $M^n = (M_s^n, \Im_s^n)$ are square integrable martingales, and $W = (W_s, \Im_s)$ are standard Wiener processes.

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Limited Theorem for Members of the Variational Series with a Random Sample Size

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Keywords: Random sample, Variational Series, Dependent Scheme, Independent Scheme.

Many publications are devoted to the study of asymptotic distributions of members of a variational series formed from independent identically distributed random variables with deterministic sample size.

In this article, we study the asymptotic distributions of the members of the variational series in the case when the sample size itself is a random variable, i.e. the characteristics of the general population under consideration are observed (due to certain circumstances) in a random number of trials. Random sample size appears in statistical problems of reliability theory, queuing theory, sequential analysis, etc. [1]. In this case, the sample size as a random variable may turn out to be independent on the observed quantities themselves (let us call this case "independent scheme"), and in some cases dependent on them (let us call this case "dependent scheme").

The results obtained present general studies of the limiting distributions of the members of the variational series with a random sample size in the "independent scheme"and "dependent scheme".

Proofs and detailed discussions of the results obtained on the asymptotic distributions of the members of the variational series with a random sample size are given in [2] and [3].

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Asymptotic approximation associated with generalized random allocation schemes

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Urn models (also known as a random allocation schemes) are a useful tool which allows to formulate and better understand many combinatorial problems in probability and statistics, see for instane, [1] and [2]. Most general definition of urn models has been introduced by Mirakhmedov et al [3], where the asymptotic theory and higher-order expansions for the statistic of the form of a sum of functions of frequencies are presented. Although their work covers many specific urn models it does not cover, such probabilistic models as, for instance, random allocation of particles into infinitely many cells, random allocation of particles in sets, and the statistics based on several samples from a population(s). The generalized random allocation scheme we are interested here is as follows.

Let $\xi_l(n_l) = (\xi_{l,1}(n_l), \xi_{l,2}(n_l), ...)$, $l = 1, ..., s$, be a collection of sequences (random vectors) of independent non-negative integer random variables (r.v.s) , distribution of $\xi_{l,m}(n_l)$ depend on parameter $n_l \in \aleph = \{1, 2, ...\}$ and such that for each n_l the series $\zeta_l(n_l) = \xi_{l,1}(n_l) + \xi_{l,2}(n_l) + ...$ is a.s. converges, moreover this parameter n_l is such that $Pr\{\zeta_l(n_l) = n_l\} > 0$,$l = 1, ..., s$. Further, let $\eta_l(n_l) = (\eta_{l,1}(n_l), \eta_{l,2}(n_l), ...)$ be a r.vec., distribution of which can be represented as the joint conditional distribution of $\xi_l(n_l)$ given $\zeta_l(n_l) = n_l$, viz.,

$$\Im(\eta_l(n_l)) = \Im(\xi_l(n_l) \mid \zeta_l(n_l) = n_l), l = 1, ..., s,$$

where $\Im(X)$ stands for the distribution of the r.vec. X. This equality implies that $\eta_{l,m}(n_l) \geq 0$, $Pr\{\eta_{l,1}(n_l) + \eta_{l,2}(n_l) + ... = n_l\} = 1$, and hence $\eta_l(n_l)$ should be viewed as r.vec. of frequencies in the random allocation of n_l particles into infinitely many cells labeled by $\{1, 2, ...\}$. The r.v.$\eta_{l,m}(n_l)$ then is the number of particles falling into the m-th cell after allocation of all n_l particles. The independent r.vec.s $\eta_1(n_1(n_1), ..., \eta_s(n_s)$ all together can be viewed, for example, as a random allocation of particles of s types, where n_l is the number of particles of the l- th type, as well as random allocation of particles in sets, where now n_l is the number of particles of l-th set. Our goal here is to present a unified approach to derive asymptotic (as $min(n_1, ..., n_s) \to \infty$) approximation for distribution function of general classes of statistics of the form

$$R(n) = \sum_{m=1}^{\infty} f_m(\eta_{1,m}, ..., \eta_{s,m}),$$

where $f_m(x_1, ..., x_s)$ is a sequence of functions (may be a random) defined for $x_1 \geq 0, ..., x_s \geq 0$, such that the series $R(n)$ is a.s. converges for every $n = (n_1, ..., n_s)$. We emphasize that the allocation scheme is determined by the distribution of r.v,s $\xi_{l,m}(n_l)$. The generalized urn model and statistics considered by Miakhmedov et al (2014) follows if $s = 1$ and $Pr\{\xi_{1,m}(n_1) = 0\} = 1$ for $m > N$. some $N = N(n) \to \infty$.

The following examples are most often encountered in applications.

(i) Independent random allocation scheme. Into an infinite number of cells,labelled by $\{1, 2, ?.\}$, we throw particles of s types, where n_k particles of k-th type, $k = 1, ..., s$. The particles are thrown randomly one by one and independently of each other. The probability of a particle of k-th type falling into m-th cell is $p_{k,m} \geq 0$ and $p_{k,1}+p_{k,2}+... = 1$. This model determines by $\Im(\xi_{l,m}(n_l)) = Poi(n_l p_{l,m})$. For instance, the number of occupied cells and the number of cells containing exactly a given number of particles of each types are examples of $R(n)$. The special case when $p_{l,m} > 0$, $m = 1, ..., N$ and $p_{l,m} = 0$ for all $m > N$, $l = 1, ..., s$, some $N = N(n_1, ..., n_s) \to \infty$ as $min(n_1, ..., n_s) \to \infty$,is well studied in the literature multinomial random allocation model, alternatively known as a sample scheme with replacement from finite population of size N. The classical chi-square, likelihood-ratio statistic, and the empty-cells statistic are examples of $R_N(n)$. Here and below $R_N(n)$ stands for the statistics $R(n)$, where $Pr\{\xi_{l,m}(n_l) = 0\} = 1$ for all $m > N$,$l = 1, ..., s$.

(ii) Sample scheme without replacement from a finite population . Assume from a stratified population of size $\Omega_N = \omega_1 + ... + \omega_N$, where $\omega_m > 0$ is the size of m-th stratum, s independent samples of sizes $n_1, ..., n_s$, respectively, are drawn. Each sample is carried out according to the sample scheme without replacement. All $C_{\Omega_N}^{n_k}$ possible variants of the kth sample has the same probability equal to $\left(C_{\Omega_N}^{n_k}\right)^{-1}$. This urn model corresponds to the case where the r.vec.$\eta_l(n_l)$ has the multivariate hyper-geometric distribution. Here $\Im(\xi_{l.m}(n_l) = Bi(\omega_m, n_l/\Omega_N)$, the binomial distribution with indicated parameters .For instance the number of untouched strata and the sample sum are important variants of $R_N(n)$. Note that the random allocation of particles in sets also is a variant of this model.

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On the identity of the theta functions

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Let Λ be an integral lattice generated by 1 and $\eta \in \mathbb{C}$, where $\operatorname{Im}\eta > 0$, and let $\mathcal{E} := \mathbb{C}/\Lambda$ be the corresponding elliptic curve. We write $\Theta_n(\Lambda)$ for the set of holomorphic functions f on $\mathbb{C}$ satisfying the quasi-periodicity conditions

$$f(z+1) = f(z)$$
$$f(z+\eta) = -e^{-2\pi i n z} f(z)\,.$$

The functions in $\Theta_n(\Lambda)$ are called *theta functions of order n* with the respect to the lattice Λ. It is well-known (see e.g. [2]), Θ_n is a vector space of dimension n. Next, one can easily check that the function

$$\theta(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi i (nz + \frac{n(n-1)}{2}\eta)}$$

forms a basis for Θ_1. For $\alpha \in \mathbb{Z}/n\mathbb{Z}$, we define

$$\theta_\alpha(z) := e^{2\pi i(\alpha z + [\alpha])} \prod_{m=0}^{n-1} \theta\left(z + \frac{m}{n} + \frac{\alpha}{n}\eta\right), \quad [\alpha] := \frac{\alpha(\alpha-n)}{2n}\eta + \frac{\alpha}{2n}. \tag{1}$$

Then $\{\theta_0, \theta_1, ..., \theta_{n-1}\}$ form a basis for Θ_n (see [1,Lemma 2.5] or [3, Appendix A]).

Let $n = 4$. Then we can rewrite (1) as

$$\theta_0(z) = \theta(z)\theta\left(z+\frac{1}{4}\right)\theta\left(z+\frac{1}{2}\right)\theta\left(z+\frac{3}{4}\right),$$
$$\theta_1(z) = \theta\left(z+\frac{1}{4}\eta\right)\theta\left(z+\frac{1}{4}+\frac{1}{4}\eta\right)\theta\left(z+\frac{1}{2}+\frac{1}{4}\eta\right)\theta\left(z+\frac{3}{4}+\frac{1}{4}\eta\right)e^{2\pi i\left(z+\frac{1}{8}-\frac{3}{8}\eta\right)},$$
$$\theta_2(z) = \theta\left(z+\frac{1}{2}\eta\right)\theta\left(z+\frac{1}{4}+\frac{1}{2}\eta\right)\theta\left(z+\frac{1}{2}+\frac{1}{2}\eta\right)\theta\left(z+\frac{3}{4}+\frac{1}{2}\eta\right)e^{2\pi i\left(2z+\frac{1}{4}-\frac{1}{2}\eta\right)},$$
$$\theta_3(z) = \theta\left(z+\frac{3}{4}\eta\right)\theta\left(z+\frac{1}{4}+\frac{3}{4}\eta\right)\theta\left(z+\frac{1}{2}+\frac{3}{4}\eta\right)\theta\left(z+\frac{3}{4}+\frac{3}{4}\eta\right)e^{2\pi i\left(3z+\frac{3}{8}-\frac{3}{8}\eta\right)}.$$

Our first main result is

Theorem 1. For any $\tau \in \mathbb{C} - \frac{1}{4}\Lambda$ the above functions satisfy the following identity

$$\frac{\theta_0^2(\tau)+\theta_2^2(\tau)}{\theta_1(\tau)\theta_3(\tau)} = \frac{\theta_1^2(\tau)+\theta_3^2(\tau)}{\theta_0(\tau)\theta_2(\tau)}. \tag{2}$$

If we introduce the following denotation

$$a := \frac{i}{\theta_0(\tau)\theta_3(\tau)},\ b := \frac{1}{\theta_0(\tau)\theta_1(\tau)},\ c := \frac{1}{\theta_2(\tau)\theta_3(\tau)},\ d := \frac{1}{\theta_0(\tau)\theta_2(\tau)}\,.$$

we can easily get the identity $a^2b^3c - b^3c^3 - a^2d^4 + b^2d^4 = 0$.

Theorem 2. The set $V := \{(a,b,c,d) \in \mathbb{P}^3 \mid a^2b^3c - b^3c^3 - a^2d^4 + b^2d^4 = 0, a \neq 0, b \neq 0, c \neq 0, d \neq 0\}$ is a normal affine algebraic surface.

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Dynamical system of an infinite-dimensional operator in an invariant set

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Denote $l^1_+ = \left\{ x = (x_1, x_2, \dots, x_n, \dots) \, : \, x_i > 0, \, \|x\| = \sum_{j=1}^{\infty} x_j < \infty \right\}$.

Following [1] we consider discrete-time, infinite-dimensional dynamical systems (IDDS) generated by operator F defined on l^1_+ as

$$F : x'_{2n-1} = \lambda_{2n-1} \cdot \left(\frac{1 + \sum_{j=1}^{\infty} x_{2j-1}}{1 + \theta + \|x\|} \right)^2, \quad x'_{2n} = \lambda_{2n} \cdot \left(\frac{1 + \sum_{j=1}^{\infty} x_{2j}}{1 + \theta + \|x\|} \right)^2$$

where $n = 1, 2, \dots$, $\theta > 0$ and $\lambda = (\lambda_1, \lambda_2, \dots) \in l^1_+$.

The main problem for such a dynamical system (see Chapter 1 of [2]) is to study trajectory $t^{(m)} = F^m(t^{(0)})$, $m \geq 1$ for any $t^{(0)} \in l^1_+$.

We have proved the following

Lemma 1. If $\lambda = (\lambda_1, \lambda_2, \dots) \in l^1_+$ then F maps l^1_+ to itself.

Define two-dimensional operator $W : z = (x, y) \in \mathbb{R}^2_+ \to z' = (x', y') = W(z) \in \mathbb{R}^2_+$ by

$$W : x' = L_1 \cdot \left(\frac{1 + x}{1 + \theta + x + y} \right)^2, \quad y' = L_2 \cdot \left(\frac{1 + y}{1 + \theta + x + y} \right)^2,$$

where $\theta > 0$ and $L_i > 0$ are parameters.

Lemma 2. The IDDS generated by the operator F is fully represented by the two-dimensional DS generated by the operator W.

Denote

$$M_- = \{(x, y) \in \mathbb{R}^2_+ : x < y\},$$
$$M_0 = \{(x, y) \in \mathbb{R}^2_+ : x = y\},$$
$$M_+ = \{(x, y) \in \mathbb{R}^2_+ : x > y\}.$$

Lemma 3. If $L_1 = L_2 = L$ then the sets M_ϵ, $\epsilon = -, 0, +$ are invariant with respect to operator W, i.e., $W(M_\epsilon) \subset M_\epsilon$.

Reduce the operator W defined by on the invariant set M_0, then we get

$$x' = f(x) := L\left(\frac{1+x}{1+\theta+2x}\right)^2. \tag{1}$$

Lemma 4. The types of fixed points are as follows
1) The unique fixed point

$$x_1 = \begin{cases} \textit{attracting}, & \text{if } \ \theta > 17, L \notin (\hat{L}_1, \hat{L}_2), \\ \textit{saddle}, & \text{if } \ \theta = 17, L = 108, \\ \textit{attarcting}, & \text{if } \ \theta = 17, L \neq 108 \ \text{ or } \ \theta \in (0,1) \cup (1,17) \end{cases}$$

2) If $\theta > 17$ and $L = \hat{L}_1$ (resp. $L = \hat{L}_2$) then the function f has two fixed points $x_1 < x_2$ and x_1 is saddle and x_2 is attracting (resp. x_1 is attracting and x_2 is saddle).
3) If $\theta > 17, L \in (\hat{L}_1, \hat{L}_2)$ then f has three fixed points with $x_1 < x_2 < x_3$. Moreover, x_1 and x_3 are attracting and x_2 is repelling;

where

$$\hat{L}_1 = \frac{2\theta^2 + 76\theta - 142 - (2\theta - 34)\sqrt{\theta^2 - 18\theta + 17}}{16},$$

$$\hat{L}_2 = \frac{2\theta^2 + 76\theta - 142 + (2\theta - 34)\sqrt{\theta^2 - 18\theta + 17}}{16}.$$

The following is main result. The following theorem describes all limit points on M_0.

Theorem 1. The following assertions hold
1) If $\theta \in (0, 17]$, $L > 0$ or $\theta > 17$, $L \notin (\hat{L}_1, \hat{L}_2)$ then for any $x \in (0, +\infty)$ the following equality holds

$$\lim_{n\to\infty} f^n(x) = x_1.$$

2) If $\theta > 17$ and $L = \hat{L}_1$ (resp. $L = \hat{L}_2$) then

$$\lim_{n\to\infty} f^n(x) = \begin{cases} x_1, & \text{if } \ x \in (0, x_1], \\ x_2, & \text{if } \ x \in (x_1, +\infty); \end{cases}$$

$$\left(\text{resp.} \quad \lim_{n\to\infty} f^n(x) = \begin{cases} x_1, & \text{if } \ x \in (0, x_2), \\ x_2, & \text{if } \ x \in [x_2, +\infty); \end{cases}\right)$$

3) If $\theta > 17, L \in (\hat{L}_1, \hat{L}_2)$ then

$$\lim_{n\to\infty} f^n(x) = \begin{cases} x_1, & \text{if } \ x \in (0, x_2), \\ x_2, & \text{if } \ x = x_2, \\ x_3, & \text{if } \ x \in (x_2, +\infty). \end{cases}$$

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Classification of non-strongly nilpotent filiform Leibniz algebras of dimension 12

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Leibniz algebra was first introduced in the early 90's of the last century by French mathematician J.-L.Lodey [2], [3] as a non-associative algebra with multiplication satisfying the identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

The class of filiform algebras are most investigated subclass of nilpotent Leibniz algebras. In [1,5] it is shown that the class of all filiform Leibniz algebras is split into three non-intersected families, where one of the families contains filiform Lie algebras and the other two families come out from naturally graded non-Lie filiform Leibniz algebras. Moreaver, an isomorphism eriterion for these two families of filiform Leibniz algebras have been given in [5]. There are series works which devoted to the classification of filiform Leibniz algebras with given dimensional. Nowadays all filiform Leibniz algebras are classified up to dimension ten (see [4,6,7]and others).In This work using the this criteria, we give the classification of non-strongly nilpotrnt Leibniz algebras for the first and the second classes of demension 12.

Definition 1. *A linear transformation P of the Leibniz algebra L is called a pre-derivation if for any $x, y, z \in L$,*

$$P([[x,y],z]) = [[P(x),y],z] + [[x,P(y)],z] + [[x,y],P(z)].$$

It is obvious that any derivation of L is a pre-derivation.

For the given Leibniz algebra L we consider the following lower central series

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1], \qquad k \geq 1.$$

Definition 2. *A Leibniz algebra L is called nilpotent if there exists $s \in N$ such that $L^s = 0$.*

Definition 3. *A nilpotent Leibniz algebra is called characteristically nilpotent if all its derivations are nilpotent. We say that a Leibniz algebra is strongly nilpotent if any pre-derivation is nilpotent.*

Definition 4. *A Leibniz algebra L is said to be filiform if $\dim L^i = n - I$, where $n = \dim L$ and $2 \leq I \leq n$.*

The following theorem divides all n-dimensional filiform Leibniz algebras into three families.

Theorem 1. *Any n-dimensional complex filiform Leibniz algebra admits a basis $\{e_1, e_2, \ldots, e_n\}$ such that the table of multiplication of the algebra has one of the following forms:*

$$F_1(\alpha_4,\dots,\alpha_n,\theta)=\begin{cases}[e_1,e_1]=e_3, & \\ [e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [e_1,e_2]=\sum\limits_{t=4}^{n-1}\alpha_te_t+\theta e_n, & \\ [e_j,e_2]=\sum\limits_{t=j+2}^{n}\alpha_{t-j+2}e_t, & 2\le j\le n-2.\end{cases}$$

$$F_2(\beta_4,\dots,\beta_n,\gamma)=\begin{cases}[e_1,e_1]=e_3, & \\ [e_i,e_1]=e_{i+1}, & 3\le i\le n-1,\\ [e_1,e_2]=\sum\limits_{t=4}^{n}\beta_te_t, & [e_2,e_2]=\gamma e_n,\\ [e_j,e_2]=\sum\limits_{t=j+2}^{n}\beta_{t-j+2}e_t, & 3\le j\le n-2,\end{cases}$$

$$F_3(\theta_1,\theta_2,\theta_3)=\begin{cases}[e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [e_1,e_i]=-e_{i+1}, & 3\le i\le n-1,\\ [e_1,e_1]=\theta_1e_n, \quad [e_1,e_2]=-e_3+\theta_2e_n, & [e_2,e_2]=\theta_3e_n,\\ [e_i,e_j]=-[e_j,e_i]\in\langle e_{i+j+1},e_{i+j+2},\dots,e_n\rangle, & 2\le i<j\le n-1,\\ [e_i,e_{n+1-i}]=-[e_{n+1-i},e_i]=\alpha(-1)^{i+1}e_n, & 2\le i\le n-1,\end{cases}$$

where all omitted products are equal to zero and $\alpha\in\{0,1\}$ for even n and $\alpha=0$ for odd n.

We give the classification of non-strongly nilpotent complex filiform Leibniz algebras of dimension 12.

Theorem 2. *Let L be a 12-dimensional non-strongly nilpotent complex filiform Leibniz algebra. Then L is isomorphic to one of the following algebras:*

$$F_1(0,1,0,0,0,\alpha_9,0,\alpha_{11},\alpha_{12},\theta), F_1(0,1,0,-3,0,0,0,\alpha_{11},\alpha_{12},\theta),$$
$$F_1(0,1,0,-3,0,12,0,0,\alpha_{12},\theta), F_1(0,1,0,-3,0,12,0,-55,1,\theta),$$
$$F_1(0,1,0,-3,0,12,0,-55,0,1), F_1(0,1,0,-3,0,12,0,-55,0,0),$$
$$F_1(0,0,0,1,0,1,0,\alpha_{11},\alpha_{12},\theta), F_1(0,0,0,1,0,0,0,1,\alpha_{12},\theta), F_1(0,0,0,1,0,0,0,-5,1,\theta),$$
$$F_1(0,0,0,1,0,0,0,-5,0,1), F_1(0,0,0,1,0,0,0,-5,0,0), F_1(0,0,0,0,0,1,0,1,\alpha_{12},\theta),$$
$$F_1(0,0,0,0,0,1,0,0,1,\theta), F_1(0,0,0,0,0,1,0,0,0,1), F_1(0,0,0,0,0,1,0,0,0,0),$$
$$F_1(0,0,0,0,0,0,0,1,1,\theta), F_1(0,0,0,0,0,0,0,1,0,1), F_1(0,0,0,0,0,0,0,1,0,0),$$
$$F_1(0,0,0,0,0,0,0,0,1,0), F_1(0,0,0,0,0,0,0,0,1,1),$$
$$F_1(1,-2,5,-14,42,-132,429,-1430,4862,0), F_1(0,0,0,0,0,0,0,0,0,0),$$
$$F_1(1,-2,5,-14,42,-132,429,0,\alpha_{12},\theta), F_1(0,0,1,0,0,-4,0,1,\alpha_{12},\theta),$$
$$F_1(1,-2,5,-14,42,-132,429,-1430,0,\theta), F_1(0,0,1,0,0,-4,0,0,0,\theta),$$
$$F_1(0,0,0,0,0,0,0,0,0,1), F_1(0,0,1,0,0,-4,0,0,22,0),$$
$$F_1(1,-2,5,-14,42,-132,429,-1430,4862,4862), F_1(0,0,1,0,0,-4,0,0,22,22),$$
$$F_1(0,0,0,0,1,0,0,1,\alpha_{12},\theta), F_1(0,0,0,0,1,0,0,0,1,\theta), F_1(0,0,0,0,1,0,0,0,0,1),$$
$$F_1(0,0,0,0,1,0,0,0,0,0), F_1(0,0,0,0,0,0,1,1,\alpha_{12},\theta), F_1(0,0,0,0,0,0,1,0,1,\theta),$$
$$F_1(0,0,0,0,0,0,1,0,0,1), F_1(0,0,0,0,0,0,1,0,0,0), F_2(0,1,0,0,0,\beta_9,0,\beta_{11},\beta_{12},1),$$

$$F_2(0,1,0,0,0,1,0,\beta_{11},\beta_{12},0), F_2(0,1,0,0,0,0,0,1,\beta_{12},0), F_2(0,1,0,0,0,0,0,0,1,0),$$

$$F_2(0,1,0,0,0,0,0,0,0,0), F_2(0,0,0,1,0,1,0,0,\beta_{12},\gamma), F_2(0,0,0,1,0,1,0,0,0,\gamma),$$

$$F2(0,0,0,1,0,0,0,0,1,\gamma), F_2(0,0,0,0,0,1,0,\beta_{11},0,1), F_2(0,0,0,0,0,1,0,1,\beta_{12},0),$$

$$F_2(0,0,0,0,0,1,0,0,1,0), F_2(0,0,0,0,0,1,0,0,0,0), F_2(0,0,0,0,0,0,0,1,0,1),$$

$$F_2(0,0,0,0,0,0,0,1,1,0), F_2(0,0,0,0,0,0,0,1,0,0), F_2(0,0,0,0,0,0,0,0,0,1),$$

$$F_2(0,0,0,0,0,0,0,0,1,0), F_2(0,0,0,0,0,0,0,0,0,0), F_2(1,0,0,0,0,0,0,1,\beta_{12},\gamma),$$

$$F_2(1,0,0,0,0,0,0,0,1,\gamma), F_2(1,0,0,0,0,0,0,0,0,1), F_2(1,0,0,0,0,0,0,0,0,0),$$

$$F_2(0,0,1,0,0,0,0,1,\beta_{12},\gamma), F_2(0,0,1,0,0,0,0,0,1,\gamma), F_2(0,0,1,0,0,0,0,0,0,1),$$

$$F_2(0,0,1,0,0,0,0,0,0,0), F_2(0,0,0,0,1,0,0,1,0,\gamma), F_2(0,0,0,0,1,0,0,1,\beta_{12},0),$$

$$F_2(0,0,0,0,1,0,0,0,0,\gamma), F_2(0,0,0,0,1,0,0,0,1,0), F_2(0,0,0,0,0,0,1,1,0,\gamma),$$

$$F_2(0,0,0,0,0,0,1,1,\beta_{12},0), F_2(0,0,0,0,0,0,1,0,0,\gamma), F_2(0,0,0,0,0,0,1,0,1,0),$$

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Extreme values of functionals of integral form

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Keywords: Extreme value, functional, convex function.

In theoretical and practical problems of mathematics and other fields of science there are problems of finding extreme values of functionals . The article is devoted to finding the exact upper bounds of the functional $Ef(\xi_1+\xi_2+...+\xi_n)$.

Let $F(x_1, x_2, ..., x_n)$ be the joint distribution of arbitrarily dependent random variables $\xi_i \in [0,1]$, $i = \overline{1,n}$. Denote by $\mathcal{F}$ the class of all distributions with fixed mathematical expectations:

$$\mathcal{F} = \{F(x_1, x_2, ..., x_n) : \ E\xi_1 = m_1, ..., E\xi_n = m_n\}.$$

Without loss of generality, we assume that $m_1 \le m_2 \le ... \le m_n$.

Theorem. Let f be the convex increasing function. Then

$$\sup_{F\in\mathcal{F}} Ef(\xi_1 + \xi_2 + ... + \xi_n) = \sum_{k=0}^{n} f(n-k)(m_{k+1} - m_k),$$

where $m_0 = 0$, $m_n = 1$. The supremum is reached on binomial distributed random variables $B_1, B_2, ..., B_n$:

$$P(B_1 = 1, B_2 = 1, ..., B_n = 1) = m_1,$$

$$P(B_1 = 0, B_2 = 0, ..., B_n = 0) = 1 - m_n,$$

$$P(B_{i_1} = 0, ..., B_{i_l} = 0, B_{i_{l+1}} = 1, ..., B_{i_n} = 0) = m_{i_{l+1}} - m_{i_l}, l = 1, 2, ..., n-1.$$

The probability values of the remaining sets 0, 1 equal to zero.

A similar statement is true when the function is f monotonically decreasing.

On the maximum of dependent random variables

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Keywords: maximum, weakly dependent random variables, limit theorems.

The distributions of the maximum of random variables have been studied by many authors. In the books [1]-[3], the results and methods for the cases of independent and weakly dependent random variables are given.

We consider stationary sequences of random variables $\{X_n, n \in N\}$ which satisfy certain conditions. We assume that there exists another sequence $\{\xi_n,\, n \in Z\}$ of random variables such that

$$X_n = f\left(\{\xi_{n+i},\ i \in Z\}\right),\ n \in N, \qquad (1)$$

where $f:\ R^Z \to R$ is a measurable function. Denote $M_n = \max\limits_{1\le i\le n} X_i$.

Our goal is to consider (1) when $\{\xi_n,\, n \in Z\}$ satisfies some weakly dependent conditions.

Namely, we assume that $\{\xi_n,\, n \in Z\}$ is $\psi-$mixing. Coefficients of $\psi-$mixing are defined as following

$$\psi\left(k\right) = \sup\left\{\left|\frac{P\left(AB\right) - P\left(A\right)P\left(B\right)}{P\left(A\right)P\left(B\right)}\right| : A \in \mathrm{F}_{-\infty}^{l},\ B \in \mathrm{F}_{l+k}^{\infty},\ l \in N,\ P\left(A\right)P\left(B\right) > 0\right\}$$

where F_a^b is $\sigma-$field algebra generated by random variables $\xi_a, \ldots, \xi_b$. A sequence of random variables $\{\xi_n, n \in Z\}$ is called satisfying the $\psi-$mixing condition, if $\psi(k) \to 0$, as $k \to \infty$.

Our aim is to establish limit distribution of $\max\limits_{1 \le i \le n} X_i$. We will assume that there exist measurable functions $f^{(m)}(\xi_{-m}, \ldots, \xi_0, \ldots, \xi_m): R^{2m+1} \to R$, $m = 1, 2, \ldots$ such that the following take place

$$\lim_{n\to\infty} nP\left(X_n^{(m)} > a_n + b_n x\right) = u(x),\ m \ge m_0 \text{ for some } m_0 \ge 0 \tag{2}$$

where $\{a_n, n \ge 1\}$, $\{b_n, n \ge 1\}$ are sequences of some constants, $X_n^{(m)} = f^{(m)}(\xi_{-m}, \ldots, \xi_0, \ldots, \xi_m)$ and $0 < u(x) < \infty$ on some interval of positive length,

$$\limsup_{n\to\infty} nP\left(\frac{1}{b_n}\left|X_1 - X_1^{(m)}\right| > \varepsilon\right) \to 0 \text{ as } m \to \infty \text{ for any } \varepsilon > 0. \tag{3}$$

Denote $M_n^{(m)} = \max\limits_{1\le i\le n} X_i^{(m)}$, $M_n = \max\limits_{1\le i\le n} X_i$.

Now we can formulate our main result.

Theorem. Let $\{X_n, n \in N\}$ be a sequence of the form (1) with a stationary $\psi-$mixing sequence $\{\xi_i, i \in Z\}$. Assume that conditions (2), (3) hold.

Then for all $x \in R$

$$P(M_n < a_n + b_n x) \to H(x) \text{ as } n \to \infty$$

where $H(x) = e^{-u(x)}$ and $e^{-\infty}$.

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Central limit theorem for ρ-mixing random variables with values in $L_p[0,1]$

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Keywords: Central limit theorem, mixing, $L_p[0,1]$ space.

Central limit theorems in Banach spaces are well studied in the case of independent identically distributed random elements (see [1]). Our goal is to establish a central limit theorem for weakly dependent random variables with values in $L_p[0,1]$ space.
We say that a sequence $\{X_i(t), i \geq 1\}$ of centered random variables in $L_p[0,1]$ satisfies central limit theorem if the following weak convergence holds:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i(t) \Rightarrow N(t)$$

where $N(t)$ is some $L_p[0,1]$-valued Gaussian random variable with mean zero.
We will assume that $\{X_n(t), n \geq 1\}$ satisfies ρ-mixing condition. For the sequence of $L_p[0,1]$-valued random variables $\{X_n(t), n \geq 1\}$ ρ-mixing coefficients are defined as:

$$\rho(n) = \sup \left\{ \frac{|E(\xi - E\xi)(\eta - E\eta)|}{E^{1/2}(\xi - E\xi)^2 E^{1/2}(\eta - E\eta)^2} : \xi \in L_2\left(F_{k+n}^{\infty}\right), \ \eta \in L_2\left(F_1^k\right), \ k \in N \right\}$$

where $\Im_a^b$-σ-field generated by random variables $X_a, ..., X_b$ and $L_2\left(F_a^b\right)$-a family of square integrable F_a^b-measurable random variables.
We say that $\{X_i(t), i \geq 1\}$ is ρ-mixing, if $\rho(n) \to 0$ as $n \to \infty$.
Our main result is the following

Theorem. Let $\{X_i(t), i \geq 1\}$ be a strictly stationary sequences of random variables with values in $L_p[0,1]$, $p \geq 2$ and the following conditions hold:

$$EX_1(t) = 0,$$

$$E|X_1(t)|^r < \infty,$$

$$\sum_{n=1}^{\infty} \rho^{\frac{2}{q}}(2^n) < \infty, \ \rho(1) < 1,$$

for some $r > p \geq q \geq 2$

$$E|X_1(t+h) - X_1(t)|^q \to 0 \text{ as } h \to 0.$$

Then $\{X_i(t), i \geq 1\}$ satisfies central limit theorem.

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Central limit theorem for a 1-order autoregressive process with random coefficients

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Keywords: *Autoregressive processes, the strong law of large numbers, the central limit theorem, Lindeberg condition.*

Let be a $\{\rho_n, n \in \mathsf{Z}\}$–sequence of independent, identically distributed random variables. A sequence $\{X_n, n \in \mathsf{Z}\}$ that satisfies equation

$$X_n - \mu = \rho_n (X_{n-1} - \mu) + \varepsilon_n, \quad n \in \mathsf{Z} \tag{1}$$

is called an autoregressive process with random coefficients, where $\mu = \mathsf{E}X_n$ and $\{\varepsilon_n, n \in \mathsf{Z}\}$–white noise: a sequence of independent identically distributed random variables with mathematical expectations equal to zero and unit variances. If the sequence $\{\rho_n\}$ satisfies condition $\sup_n |\rho_n| < 1$ and does not depend on sequence $\{\xi_n\}$, then there exists a unique strictly stationary solution of equation (**??**) such that $\mathsf{E}X_n = \mu$. This solution has the form

$$X_n = \mu + \sum_{k=0}^{\infty} A_{nk}\varepsilon_{n-k}, \quad n \in \mathsf{Z} \tag{2}$$

where

$$A_{nk} = \prod_{j=n}^{n-k+1} \rho_j, \quad k \geq 1, \quad A_{n0} = 1$$

and the series converges a.s.

The paper [1] considers a class of autoregressive processes of the 1–order with random coefficients taking values in the Hilbert space. Limit theorems for this class are obtained: the strong law of large numbers, the central limit theorem (c.l.t.), the compact law of the iterated logarithm. exponential inequalities, as well as rates of convergence.

In this paper, we consider an autoregressive process of the 1–order with random coefficients, which is not included in the class from [1]:

$$X_n = \nu X_{n-1} + \xi_n, \quad n \in \mathsf{Z} \tag{3}$$

where $0 < \nu < 1$–random variable independent of the innovation sequence $\{\xi_k\}$. There exists (see, for example, [2]) a unique strictly stationary solution of equation (3) such that $\mathsf{E}X_n = 0$. This solution has the form

$$X_n = \sum_{k=0}^{\infty} \nu^k \varepsilon_{n-k}, \quad n \in \mathsf{Z}$$

Let $X_k = \sum\limits_{j=0}^{\infty} A_j \varepsilon_{k-j}$, $n \in \mathsf{Z}$–linear process with random coefficients $\{A_k,\ k = 0, 1, \dots\}$ generated by a sequence $\{\xi_l,\ l \in \mathsf{Z}\}$ of independent random variables and $\sum\limits_{k=1}^{n} X_k$. We use the following decomposition of the linear process X_n.

Lemma 1. If the series $\sum\limits_{k=0}^{\infty} A_k$ converges absolutely, then decomposition

$$X_n = \left(\sum_{j=0}^{\infty} A_j\right)\xi_n + \sum_{j=1}^{\infty}\gamma_j\xi_{n-j} - \sum_{j=1}^{\infty}\gamma_j\xi_{n-j+1} \tag{4}$$

takes place, where $\gamma_j = \sum\limits_{k=j}^{\infty} A_k$.

Equality (4) can be proved directly by comparing the coefficients in front of the random variables ξ_l; $l = 0, \pm 1, \pm 2, \ldots$ in the expression of the random variable X_n. In particular, from expansion (4) we get the following expansion for the sum of the first n terms of the linear process:

$$S_n = \left(\sum_{j=0}^{\infty} A_j\right)\sum_{t=1}^{n}\xi_t + \sum_{j=1}^{\infty}\gamma_j\left(\xi_{n-j} - \xi_{n-j+1}\right) \tag{5}$$

Using expansion (5), in this paper we obtain a proof of the following theorem.

Theorem 1. Let $X_k = \sum\limits_{j=0}^{\infty} \nu^j \xi_{k-j}$ a linear process with random coefficients satisfies conditions

1) $A_k = \nu^k$, $0 < \nu < 1$. Random variable ν does not depend on $\{\xi_k, k \in \mathsf{Z}\}$;

2) $\{\xi_k\}$ a sequence of independent random variables, $\mathsf{E}X_k = 0$, $\mathsf{E}X_k^2 = \sigma_k^2 < \infty$, $k \geq 0$;

3) $\max\limits_{0\leq k\leq n} \dfrac{\sigma_k^2}{B_n^2} \to 0$ as $n \to \infty$, where $B_n^2 = \sum\limits_{k=1}^{n} \sigma_k^2$.

Then the sequence $(1-\nu)S_n\big/B_n$ satisfies the c.l.t. if and only if the Lindeberg condition is satisfied:

$$\frac{1}{B_n^2}\sum_{k=1}^{n}\mathsf{E}\xi_k^2 I\left\{|\xi_k \geq \varepsilon|\right\} \longrightarrow 0 \quad \text{as} \quad n\to\infty \quad \text{for any positive} \quad \varepsilon. \tag{L}$$

Remark 1. Let $\nu \in (0,1)$ and $X_k = \sum\limits_{j=0}^{\infty} \nu^j \xi_{k-j}$ a linear process with constant coefficients satisfies conditions 2) and 3). Then the sequence $(1-\nu)S_n\big/B_n$ satisfies the c.l.t. if and only if the Lindeberg condition is satisfied:

$$\frac{1}{B_n^2}\sum_{k=1}^{n}\mathsf{E}\xi_k^2 I\left\{|\xi_k \geq \varepsilon|\right\} \longrightarrow 0 \quad \text{as} \quad n\to\infty \quad \text{for any positive} \quad \varepsilon. \tag{L}$$

In the following theorem, we obtain an estimate for the rate of convergence in the c.l.t. for more general linear processes with random coefficients.

Theorem 2. Let $X_k = \sum_{j=0}^{\infty} A_j\xi_{k-j}$ a linear process with random coefficients $\{A_k,\ k = 0, 1, \ldots\}$, generated by a sequence of $\{\xi_l, l \in \mathsf{Z}\}$ independent random variables and $S_n = \sum_{k=1}^{n} X_k$. If conditions are satisfies

1) $\mathsf{E}A_j^2 = a_j^2 < \infty$; $A = \sum\limits_{j=0}^{\infty} A_j$, $|A| > 1$; $\mathsf{E}\left|\sum\limits_{k=1}^{\infty} kA_k\right|^s = b_s < \infty$;

2) $\mathsf{E}\xi_k = 0$, $\mathsf{E}\left|\xi_k\right|^s = \beta_{sk} < \infty$;

3) $\{A_j\}$ does not depend on sequence $\{\xi_k\}$, then the following estimate holds:

$$\triangle_n = \sup_x \left| P\left(\frac{S_n}{AB_n \leq x} - \Phi(x)\right)\right| \leq C(s)L(s) + 2^{\frac{2s-1}{s+1}} \pi^{\frac{1}{s+1}} \frac{b_s^{\frac{1}{s+1}} \left(\beta_{s0} + \beta_{sn}\right)^{\frac{1}{s+1}}}{B_n^{s/(s+1)}}.$$

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